Assignment 5

1. Prove that the Fourier transform of a Schwartz function, \( f \in \mathcal{S}(\mathbb{R}^d) \), is a Schwartz function.

2. Area and volume of the unit sphere. You can skip this exercise if you know it already, but it is otherwise a must-do!

   Let \( |S_d| \) be the area of the \( d \)-dimensional unit sphere (that is, the surface is \( d \)-dimensional), and \( |B_d| \) the volume of the \( d \)-dimensional ball.

   (a) Show that
   \[
   |S_d| = \frac{2\pi^{d+1}}{\Gamma\left(\frac{d+1}{2}\right)},
   \]
   where \( \Gamma(s) = \int_0^\infty e^{-t}t^{s-1}dt \) is the Gamma function.

   Hint: Calculate \( \int_{\mathbb{R}^d} e^{-\pi \|x\|^2}dx = 1 \) using polar coordinates.

   (b) Show that \( d|B_d| = |S_{d-1}| \), hence
   \[
   |B_d| = \frac{\pi^{d/2}}{\Gamma\left(\frac{d}{2} + 1\right)}.
   \]

3. Heat equation, formal calculations. Consider the heat equation in \( \mathbb{R}^d \):
   \[
   \frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x_1^2} + \cdots + \frac{\partial^2 u}{\partial x_d^2},
   \]
   for \( t > 0 \), with initial value \( u(x,0) = f(x) \) and \( f \in \mathcal{S}(\mathbb{R}^d) \).

   Take the Fourier transform of the heat equation, and solve it for any fixed \( k \). Show that the formal solution can be written
   \[
   u(x,t) = \int_{\mathbb{R}^d} \hat{f}(k) e^{-4\pi^2t\|k\|^2} e^{2\pi ikx} dk.
   \]

4. Heat equation, rigorously. Define the heat kernel \( H_t(x) \) by
   \[
   H_t(x) = \frac{1}{(4\pi t)^{d/2}} e^{-\|x\|^2/4t} = \int_{\mathbb{R}^d} e^{-4\pi^2t\|k\|^2} e^{2\pi ikx} dk.
   \]
   We consider the function \( u(x,t) = (f * H_t)(x) \).
(a) Show that $u(x, t)$ is equal to the formal solution of Exercise 3.

(b) Show that $u(x, t)$ is a Schwartz function in $x$, for each fixed $t > 0$.

(c) Show that $u(x, t)$ solves the heat equation for each $t > 0$.

(d) Show that $\lim_{t \to 0^+} u(x, t) = f(x)$, uniformly in $x$. 