Gibbs phenomenon

Consider the sign function on $\mathbb{T}$, which is the simplest discontinuous function:

$$\text{sgn}(t) = \begin{cases} 1 & \text{if } 0 < t < \frac{1}{2} \\ 0 & \text{if } t = 0 \text{ or } t = \frac{1}{2} \\ -1 & \text{if } -\frac{1}{2} < t < 0 \end{cases}$$

We know from Jordan’s criterion (Theorem 2.2) that $S_N \text{sgn}(x) \rightarrow \text{sgn}(x)$ as $N \rightarrow \infty$, for all $x \in \mathbb{T}$. One can also show that $\|S_N \text{sgn} - \text{sgn}\|_p \rightarrow 0$ for all $1 \leq p < \infty$.

However,

$$\lim_{N \rightarrow \infty} \sup_{x \in \mathbb{T}} |S_N \text{sgn}(x)| = \frac{2}{\pi} \int_0^{1/2} \frac{\sin t}{t} \, dt = 1.17898...$$

Then $\|S_N \text{sgn} - \text{sgn}\|_\infty \rightarrow 0$. The reason is that $S_N \text{sgn}$ looks like that:

![Diagram of Gibbs phenomenon]

This is known as Gibbs phenomenon, and it is important in image processing.

3. Fourier series of $L^p([0,1])$ functions

Let us recall that $L^p([0,1]) = L^p([0,1], \mathbb{R})$, so that $S_n f$ is well-defined for any $f \in L^p([0,1])$. We consider the two questions:

1. Does $\lim_{n \rightarrow \infty} \|S_n f - f\|_p = 0$ for every $f \in L^p([0,1])$?

2. Does $\lim_{n \rightarrow \infty} S_n f(x) = f(x)$ a.e. if $f \in L^p([0,1])$?

Here is a result that helps to answer the first question.

Lemma 3.1

Let $p \in [1,\infty)$. Then $\|S_n f - f\|_p \rightarrow 0$ if and only if there exists $C_p$ such that

$$\|S_n f - f\|_p \leq C_p \|f\|_p$$

for all $n \geq 1$, all $f \in L^p([0,1])$.

Proof: We first prove $\Rightarrow$. If $g$ is a trigonometric polynomial, then $S_n g = g$ for all $n \geq \deg g$. Since trigonometric polynomials are dense in $L^p$, $\forall \varepsilon > 0$ there exists a trig. pol. $g$ such that $\|f - g\|_p \leq \varepsilon$. Then

$$\|S_n f - f\|_p \leq \|S_n (f - g)\|_p + \|S_n g\|_p + \|g - f\|_p \leq (C_p + 1) \varepsilon.$$
provided $n \geq \deg y$. This proves $\Delta = 0$.

The converse claim follows from the uniform boundedness theorem. (If $\|S_n f - \tilde{f}\|_p \to 0$, then $\|S_n f\|_p \leq C$ for some constant $C$ depending on $f$, not on $n$.)

According to Lemma 3.1, we have $\|S_n f - \tilde{f}\|_p \to 0$ if the operator norm of $S_n : L^p(\mathbb{R}^d) \to L^p(\mathbb{R}^d)$ is bounded uniformly in $n$. For $p = 1$,

$$
\|S_n\| = \sup_{f \in L^1} \frac{\int_0^1 \int_{\mathbb{R}^d} |D_n f(t,x)| dx \, dt}{\|f\|_1} = \sup_{f \in L^1} \left( \frac{\int_0^1 \int_{\mathbb{R}^d} |D_n f(t,x)| dx \, dt}{\|f\|_1} \right) = \frac{\|D_n\|}{\|D\|}.
$$

Furthermore, using $f \approx S_0$, we can show that

$$
\|S_n\| \geq \frac{\int_0^1 dt \int_{\mathbb{R}^d} |D_n f(t,x)| dx \, dt}{\|f\|_1} = \frac{\int_0^1 \|D(t)\| dx \, dt}{\|f\|_1} = \frac{\|D_n\|}{\|D\|}.
$$

We saw in the previous chapter that $\|D\| = \frac{2}{\pi} \log n$. By Lemma 3.1, we can conclude that it is not true that $\|S_n f - \tilde{f}\|_p \to 0 \forall f \in L^1$.

We shall see that $\|S_n f - \tilde{f}\|_p \to 0$ for all $1 \leq p < \infty$. And it is not true for $p = \infty$, since the limit of $S_n f$ is continuous if it exists, but there exist discontinuous functions in $L^\infty(\mathbb{R}^d)$.

Theorem 3.2

Let $p \in (1, \infty)$. Then

(a) If $f \in L^p(\mathbb{R}^d)$, then $\lim_{n \to \infty} \|S_n f - \tilde{f}\|_p = 0$.

(b) The trigonometric polynomials $(\sum_{k \in \mathbb{Z}^d} c_k e^{2\pi i x \cdot k})$ are dense.

(c) If $f \in L^1(\mathbb{R}^d)$ and $f(k) = 0 \forall k \in \mathbb{Z}^d$, then $f = 0$.

Let us recall Minkowski inequality: $\|f + g\|_p \leq \|f\|_p + \|g\|_p$, $1 \leq p < \infty$, and more generally: If $\mu, \nu$ are $\sigma$-finite measures on $\mathbb{R}^d$, then $\langle f, \mu \rangle$ is a non-negative measurable function on $\mathbb{R}^d$, then

$$
\left( \int_{\mathbb{R}^d} |f(x)|^p \mu(dx) \right)^{\frac{1}{p}} \leq \int_{\mathbb{R}^d} |f(x)|^p \mu(dx).
$$

Proof of Theorem 3.2: Recall that $\tilde{f} = F \ast \tilde{f}$, where $F$ is the Fejér kernel $F_n(t) = \frac{1}{n} \sin^2(\pi nt)$. We have

$$
\|S_n f - \tilde{f}\|_p = \left( \int_0^1 dt \int_{\mathbb{R}^d} |\sup_{s \in [0,t]} |F_n(s)| \left( F_n(s) - F(s) \right) |^p \right)^{\frac{1}{p}} \leq \int_0^1 dt \int_{\mathbb{R}^d} |F_n(s)| \|F(s) - \tilde{f}\|_p.
$$

Using the bounds $\|F_n(s)\| = \frac{2}{\pi} \log n$,

$$
\|S_n f - \tilde{f}\|_p \to 0 \quad \text{as} \quad n \to \infty.
$$
For every $\varepsilon > 0$, there exists $g \in C([0,1])$ such that $\|f - g\|_p < \frac{\varepsilon}{3}$, so that
\[
\|f(x) - g(x)\|_p \leq \|f(x) - g(x)\|_p + \|g(x) - g(x)\|_p + \|g(x) - f(x)\|_p < \varepsilon,
\]
provided that $\varepsilon$ is small enough. Then $\|\alpha_n - f\|_p$ is arbitrarily small if $n$ is large. This proves (a).

(b) follows since $\alpha_n f$ is a trigonometric polynomial (why? — Write the details!). (c) is also immediate since $\hat{f}(1) \equiv 0 \implies \alpha_n f \equiv 0$ for all $n$.

$p=2$ is a special case because $L^2([0,1])$ is a Hilbert space.

The functions $(e^{i2\pi kn})_{k \in \mathbb{Z}}$ form an orthonormal set:

$$\langle e^{i2\pi kn}, e^{i2\pi lm} \rangle = \sum_{k \in \mathbb{Z}} e^{i2\pi (kn - lm)} = \frac{1}{2\pi i (l-k)} e^{i2\pi (k-l)} 1_{k \neq l} = 0.$$

By Theorem 3.2 (b), the functions $(e^{i2\pi kn})_{k \in \mathbb{Z}}$ form an orthonormal basis. The Fourier coefficient $\hat{f}(k)$ is the projection of $f$ onto $e^{i2\pi kn}$. By the properties of orthonormal bases, we have

$$\|f\|_2^2 = \sum_{k \in \mathbb{Z}} |\hat{f}(k)|^2.$$

This is Parseval or Plancherel theorem. It implies that the second property of Lemma 3.1 is true with constant $C_2=1$.

We should also note that the Fourier series can be seen as a unitary operator from $L^2([0,1])$ to $\ell^2(\mathbb{Z})$.

**Theorem 3.3**

For every $p \in (1, \infty)$, for every $f \in L^p([0,1])$, we have

$$\|\alpha_n f - f\|_p \to 0$$

as $n \to \infty$.

The proof is rather complicated and is omitted. An option is to consider the “Hilbert transform” $H : L^p \to L^p$ where

$$Hf(k) = \frac{1}{\pi} \text{P.V.} \hat{f}(k).$$

One can prove that $\|H\| < \infty$ (Riesz), which implies that $C_p < \infty$ in Lemma 3.1.

We conclude the chapter with two applications of Fourier series: the isoperimetric inequality, and Weyl ergodic theorem. In both cases, Fourier theory offers surprising and rather simple proofs.
Isoperimetric inequality ("Dido's problem")

This is a surprising consequence of Fourier series. According to the legend, Princess Elissa of Tyre had to flee her native city, and she arrived at the location of the future Carthage. She asked the Berber king Tarboq for a small piece of land, just as much as could be encompassed by an oxhide. Tarboq agreed, and Elissa cut the oxhide in four strips so she could encircle an entire hill. Princess Elissa became Queen Dido and the hill became Carthage. This took place around 800 B.C.

The question is what is the maximal area with given length? Or the minimal length that contains a given area? Clearly, the answer is the circle:

\[ \bigcirc \]

**Theorem 3.4 (Isoperimetric inequality)**

Let \( P \) be a simple closed curve in \( \mathbb{R}^2 \) of length \( L \), and \( A \) be the area of the region enclosed by the curve. Then

\[ A \leq \frac{L^2}{4\pi}, \]

with equality iff \( P \) is a circle.

**Proof:** We can assume that \( L = 1 \) by rescaling. Let \( (x(t), y(t)) \) be a parametrization of the curve \( P \), with \( t \in [0, 1] \) and

\[ x'(t)^2 + y'(t)^2 = 1. \]

Then

\[ \int_0^1 (x'(t)^2 + y'(t)^2) \, dt = 1. \]

Since \( \int_0^1 x'(t) \, dt = 2\pi \text{ length of } P \), and similarly for \( y \), we have from Plancherel's theorem

\[ 4\pi \sum_{k \in \mathbb{Z}} |k| \left( |\hat{x}(k)|^2 + |\hat{y}(k)|^2 \right) = 4\pi \]

Next, we need an expression for the area in terms of \((x(t), y(t))\).

We could use Green's theorem, but we can also derive the expression

\[ \frac{(x(0), y(0)) + (x(\pi), y(\pi))}{2} = \frac{\pi}{2} \]

\[ \text{area} = \frac{1}{2} \left( \int_0^1 (x(t), y(t)) \, dt = \frac{1}{2} \right. \]

\[ \left. \int_0^1 (x(t), y(t)) \, dt = \frac{1}{2} \left( x(0)y(0) - x(\pi)y(\pi) \right) \right) \]

\[ A = \frac{1}{2} \int_0^1 (x(t)y'(t) - x'(t)y(t)) \, dt = \pi i \sum_{k \in \mathbb{Z}} k \left( \hat{x}(k) \hat{y}(k) - \hat{x}(k) \hat{y}(k) \right) \]

We used Parseval's identity, \( \int_0^1 f(t) g(t) \, dt = \sum_{k \in \mathbb{Z}} \hat{f}(k) \hat{g}(k) \), observing that \( x(t) \) and \( y(t) \) are real. Then