Assignment 2

1. Prove that the Fejér kernel "converges to Dirac". Precisely, show that $F_n(t)=\frac{1}{n}\frac{\sin^2(n\pi t)}{\sin^2(\pi t)}$ satisfies

- (a) $F_n(t) \ge 0$. (Obvious!)
- (b) $\int_{-\frac{1}{2}}^{\frac{1}{2}} F_n(t) dt = 1.$
- (c) $\lim_{n\to\infty} \int_{|t|>\delta} F_n(t) dt = 0$ for any $\delta > 0$.

2. Properties of convolution. Suppose that $f,g,h\in L^1([0,1])$. Prove the following facts:

- (a) f * (g+h) = f * g + f * h.
- (b) (cf) * g = c(f * g) = f * (cg) for any $c \in \mathbb{C}$.
- (c) f * g = g * f.
- (d) (f * g) * h = f * (g * h).
- (e) f * g is continuous.
- (f) $\widehat{f*g}(k) = \widehat{f}(k)\,\widehat{g}(k)$.

3. Abel summability. Let $f \in L^1([0,1])$, and with $r \in (0,1)$, define

$$A_r f(x) = \sum_{k \in \mathbb{Z}} r^{|k|} \widehat{f}(k) e^{2\pi i k x}.$$

- (a) Prove that this series converges absolutely for all $x \in [0, 1]$, and that $A_r f(x)$ is a continuous function of x.
- (b) Let P_r be the Poisson kernel such that $A_r f = P_r * f$. Show that

$$P_r(x) = \sum_{k \in \mathbb{Z}} r^{|k|} e^{2\pi i k x} = \frac{1 - r^2}{1 - 2r \cos(2\pi x) + r^2}.$$

- (c) Prove that if $f \in L^p([0,1])$, $1 \le p < \infty$, we have $\lim_{r \nearrow 1} ||A_r f f||_p = 0$. (This should be similar to the proof of Theorem 3.2 (a).) What about $p = \infty$?
- **4.** Show that the Fourier series of a periodic differentiable function $f \in C^1(\mathbb{T})$ is absolutely convergent. (*Hint:* Use the Cauchy-Schwarz inequality and Parseval formula for f'.)
- **5.** Let $f \in C^{\alpha}(\mathbb{T})$, with $\alpha \in \mathbb{N}$. Show that $\widehat{f}(k) = o(1/|k|^{\alpha})$.