The inverse Fourier transform \( \hat{F}' \) of \( F \in L^1 \) is
\[
\hat{F}'(\omega) = \frac{1}{(2\pi)^d} \int e^{i\omega \cdot \xi} f(\xi) \, d\xi.
\]
If \( F, \hat{F} \) both belong to \( L^1 \), we have \( F = (\hat{F}')' = \hat{F}'' \). Further, we have Plancherel theorem:
\[
\|F\|_2 = \frac{1}{(2\pi)^d} \|\hat{F}\|_2.
\]
(Unless specified otherwise, \( \| \cdot \|_2 \) denotes the \( L^2 \)-norm.) The theorem also applies to the inner product:
\[
(F, g) = \frac{1}{(2\pi)^d} (\hat{F}, \hat{g}).
\]
Recall that the Fourier transform of \( L^1 \) functions, defined above, can be extended to all \( L^2 \) functions by \( \ell^2 \)-continuity. Indeed, let \( F : L^1 \to L^2 \) denote the map \( F \mapsto \hat{F} \). It is defined on a dense set of functions in \( L^2 \) by the integral above. It is linear, and \( \|F\| = (2\pi)^d \|\hat{F}\|_2 \leq \infty \) so \( F \) is continuous and it can be extended to \( L^1 \) by \( L^2 \). The inverse map \( F^{-1} \) exists for the same reason.

**Definition 1.** A function \( f \in C_c^\infty (\mathbb{R}^d) \) is Schwartz if \( \forall k \in \mathbb{N}^d \), \( \forall \lambda \in \mathbb{N} \), we have
\[
\sup_{x \in \mathbb{R}^d} \|x^k \partial_x^\lambda f(x)\| < \infty.
\]

The Fourier transform plays a central role in quantum mechanics.

**Definition 2.** The Fourier transform \( \hat{f} \) of \( f \in L^1(\mathbb{R}^d) \) is
\[
\hat{f}(k) = \int e^{i\omega \cdot x} f(x) \, dx.
\]
If this function exists, it is unique.

**Definition 4.** The Sobolev space $H^s = H^s(\mathbb{R}^d)$ is the space of all $f \in L^2$ such that $\frac{df}{dx}$ exists and belongs to $L^2$, for all $j=1, \ldots, d$. It is a Hilbert space with the Sobolev inner product

$$\langle f, g \rangle_{H^s} = \langle f, g \rangle_{L^2} + \sum_{j=1}^{d} \left( \frac{df}{dx} \cdot \frac{d^j g}{dx^j} \right)_{L^2} = \frac{1}{(2\pi)^{d/2}} \int_{\mathbb{R}^d} \left( 1 + |x|^2 \right)^{s/2} \overline{f(x)}g(x) \, dx.$$

II. Operator theory

We review essential notions about operators in infinite-dimensional Hilbert spaces. The Laplacian is an interesting and useful example. The main complication of unbounded operators is that they are only defined on a subspace of the Hilbert space.

**Definition 5.** A densely-defined operator on $H$ is a pair $(T, D(T))$ where the domain $D(T)$ is a dense linear subspace of $H$, and $T$ is a linear map $D(T) \to H$.

For example, $H = L^2(\mathbb{R}^d)$, $T = \Delta = \sum_{j=1}^{d} \frac{d^2}{dx_j^2}$, and $D(T) = C_c^\infty(\mathbb{R}^d)$ (the Schwartz space), or $D(T) = L^2 \cap C^2$, ... Exercise: check that $\Delta$ is unbounded.

Any operator $T$ has an adjoint $T^*$. In words, $T^*$ is an operator $D(T^*) \to H$ such that $(T^*f, g) = (f, Tg)$ for all $f \in D(T)$ and $g \in D(T^*)$, and the domain of $T^*$ is the largest possible. Notice that $D(T^*)$ is not necessarily dense.

**Definition 6.** Let $T: D(T) \to H$, where $D(T)$ is closed in $H$. The domain of the adjoint is

$$D(T^*) = \left\{ f \in H : \exists g \in D(T) \text{ such that } (T^*f, g) = (f, Tg) \text{ for all } g \in D(T) \right\}.$$

If such $g$ exists, it is unique because $D(T)$ is dense. Then the adjoint is the operator $T^*$ that assigns $T^*f = g$ to each $f \in D(T^*)$.

In the case where $T$ is bounded, Riesz representation theorem implies that $D(T^*) = H$. And if $\dim H < \infty$, the adjoint of a matrix is the hermitian conjugate.

**Definition 7.** An operator $T: D(T) \to H$ is symmetric if $(Tf, g) = (f, Tg)$ for all $f, g \in D(T)$.

An operator $T: D(T) \to H$ is self-adjoint if $T^* = T$.

One can check that $T$ is symmetric if $T^*$ is an extension of $T$, i.e., if $D(T^*) \supset D(T)$ and $T^*f = Tf$ for all $f \in D(T)$. 

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