It is usually easy to verify that an operator is symmetric, harder that it is self-adjoint.

Example: Laplacian with different boundary conditions.
Let $\mathcal{D} = L^2((0,1))$ and $T_k = \frac{d^k}{dx^k}$, $k=1,2,3,4$ with the
following domains:
\[
\begin{align*}
\mathcal{D}(T_1) &= \{ u \in C^2([0,1]) : u(0) = u(1) = 0 \} \\
\mathcal{D}(T_2) &= C^2([0,1]) \\
\mathcal{D}(T_3) &= \{ u \in H^2([0,1]) : u(0) = u(1) = 0 \} \\
\mathcal{D}(T_4) &= H^4([0,1])
\end{align*}
\]

For any $f, g \in H^4$, we have
\[
\begin{align*}
\langle f, \frac{d^4}{dx^4} g \rangle &= \int_0^1 f(x) \frac{d^4}{dx^4} g(x) \, dx = \left[ \frac{d^2}{dx^2} f(x) \frac{d^2}{dx^2} g(x) \right]_0^1 - \int_0^1 \frac{d^2}{dx^2} f(x) \frac{d^2}{dx^2} g(x) \, dx \\
&= \left[ f(x) \frac{d}{dx} g(x) \right]_0^1 - \int_0^1 f(x) \frac{d}{dx} g(x) \, dx + \left( \frac{d^2}{dx^2} f(x), g(x) \right)
\end{align*}
\]
This shows that $T_1$ and $T_2$ are symmetric, $T_3$ and $T_4$ are not symmetric. $\mathcal{D}(T_k)$ contains $\mathcal{D}(T_j)$ but it is bigger, since it also contains $f \in H^2([0,1])$ with $f(0) = f(1) = 0$.
It turns out that $T_1 = T_5$ and $T_2 = T_6$. $T_2$ is self-adjoint, it is the Laplacian with Dirichlet boundary conditions.
$T_3$ and $T_4$ are not self-adjoint since they are not symmetric.

The spectrum of operators in infinite-dimensional spaces is more complicated than that of matrices, which consists of a finite number of eigenvalues. The complex plane is now the disjoint union of the resolvent set, the point spectrum, the continuous spectrum, and the residual spectrum:

- The resolvent set $\rho(T)$ of the operator $T$ is the set of complex numbers $\lambda$ such that $T-\lambda I$ is one-to-one, and the inverse operator $(T-\lambda I)^{-1}$ is bounded.
- The point spectrum $\sigma_p(T)$ is the set of $\lambda \in \mathbb{C}$ such that $T-\lambda I$ is not one-to-one. That is, there exists $f \in \mathcal{D}(T)$ such that $(T-\lambda I)f = 0$, or $Tf = \lambda f$.
- The continuous spectrum $\sigma_c(T)$ is the set of $\lambda \in \mathbb{C}$ such that $T-\lambda I$ is one-to-one, and the range of $(T-\lambda I)^{-1}$ is dense, but $(T-\lambda I)^{-1}$ is unbounded.
- The residual spectrum $\sigma_r(T)$ is the set of $\lambda \in \mathbb{C}$ such that $T-\lambda I$ is one-to-one and $\text{ran}(T-\lambda I)$ is not dense.

Theorem 1. If $T$ is self-adjoint, $\sigma_p(T) = \emptyset$, $\sigma_c(U) = \mathbb{R}$, and $\sigma_r(T)$ is closed.
We now construct the evolution operator $U_t = e^{-it\mathbf{A}}$ for $\mathbf{A}$ self-adjoint but possibly unbounded. This is not immediate and we must do it for bounded operators.

**Theorem 2**

Let $\mathbf{A}$ be a bounded operator on $\mathcal{H}$. Then

(a) The sequence of operators $\left(\sum_{n=0}^{M} \frac{(-it)^n}{n!}\mathbf{A}^n\right)$ is Cauchy.

The space of bounded operators on $\mathcal{H}$ is Banach, so that the following operator is well-defined for any fixed $t \in \mathbb{R}$:

$$U_t = \sum_{n=0}^{\infty} \frac{(-it)^n}{n!}\mathbf{A}^n = e^{-it\mathbf{A}}.$$

(b) For any $t \in \mathbb{R}$, $U_{it} U_t = U_{-t} U_t$.

(c) $\frac{d}{dt} U_t = -i \mathbf{A} U_t$.

Let us assume in addition that $\mathbf{A}$ is self-adjoint. Then

(d) $U_t^* = U_{-t}$.

(e) $U_t$ is unitary.

**Proof:** For (a), observe that, if $M < N$,

$$\sum_{n=0}^{M} \frac{(-it)^n}{n!}\mathbf{A}^n - \sum_{n=0}^{M} \frac{(-it)^n}{n!}\mathbf{A}^n = \sum_{n=M+1}^{\infty} \frac{(-it)^n}{n!}\mathbf{A}^n.$$

The norm of the left side is then less than $\sum_{n=M+1}^{\infty} \frac{1}{n!} \|\mathbf{A}\|^n$, which is small when $M, N$ are large. This proves (a).

For (b), we have

$$U_0 U_t = \sum_{m=0}^{\infty} \frac{(-it)^m}{m!}\mathbf{A}^m = \sum_{k=0}^{\infty} \frac{1}{k!}\mathbf{A} \sum_{n=0}^{k} \frac{1}{n!} \mathbf{A} \sum_{m=0}^{n} \frac{(-it)^n}{n!} \mathbf{A}^n = U_{it} U_t.$$

(c) is also easy, since all series converge:

$$\frac{d}{dt} U_t = \lim_{\varepsilon \to 0} \frac{U_{t+\varepsilon} - U_t}{\varepsilon} = \lim_{\varepsilon \to 0} \sum_{n=0}^{\infty} \frac{(-i\varepsilon)^n}{n!} \mathbf{A}^n = -i \mathbf{A} U_t.$$

(d) follows from the Taylor series. And (e) follows from (d), since $U_t^* = U_{-t}$, hence $U_{it}^* = iU_t$, since $U_{it} U_t = \text{Id}$, hence for $U_t U_{it}^* = \text{Id}$. □

**Lemma 2.** Let $\mathbf{A}$ be a self-adjoint operator. For any $S \in \mathbb{R} \setminus \{0\}$,

$$\| (\mathbf{A} + iS)^{-1} \| \leq \frac{1}{|S|}.$$