It is usually easy to verify that an operator is symmetric, harder that it is self-adjoint.

<u>Example</u>: Laplacian with different boundary conditions.

Let $\mathcal{H} = L^2([0,1])$ and $T_k = \frac{d^2}{dx^2}$, $k = 1, 2, 3, 4$ with the following domains:

$$D(T_1) = \{ u \in C^2([0,1]) : u(0) = u(1) = 0 \}$$
$$D(T_2) = C^2([0,1])$$
$$D(T_3) = \{ u \in H^2([0,1]) : u(0) = u(1) = 0 \}$$
$$D(T_4) = H^2([0,1])$$

For any $f, g \in H^2$, we have

$$(f, \tfrac{d^2}{dx^2} g) = \int_0^1 \overline{f(x)} \, \frac{d^2 g}{dx^2}(x) \, dx = \overline{f(x)} \frac{dg}{dt}(x) \Big|_0^1 - \int_0^1 \frac{\overline{df}}{dx} \frac{dg}{dx} \, dx$$
$$= \overline{f(x)} \frac{dg}{dt}(x) \Big|_0^1 - \frac{\overline{df}}{dx}(x) \, g(x) \Big|_0^1 + \left(\tfrac{d^2}{dx^2} f, g \right).$$

This shows that T_1 and T_3 are symmetric, T_2 and T_4 are not symmetric. $D(T_1^*)$ contains $D(T_1)$ but it is bigger, since it also contains $f \in H^2([0,1])$ with $f(0) = f(1) = 0$.
It turns out that $T_1^* = T_3$ and $T_3^* = T_3$. T_3 is self-adjoint, it is the Laplacian with Dirichlet boundary conditions.
T_2 and T_4 are not self-adjoint since they are not symmetric.

The spectrum of operators in infinite-dimensional spaces is more complicated than that of matrices, which consists of a finite number of eigenvalues. The complex plane is now the disjoint union of the resolvent set, the point spectrum, the continuous spectrum, and the residual spectrum:

- The <u>resolvent set</u> $\rho(T)$ of the operator T is the set of complex numbers λ such that $T - \lambda \, \mathrm{Id}$ is one-to-one, $\operatorname{ran}(T - \lambda \, \mathrm{Id})$ is dense in $\mathcal{H}$, and the inverse operator $(T - \lambda \, \mathrm{Id})^{-1} : \operatorname{ran}(T - \lambda \, \mathrm{Id}) \to D(T)$ is bounded.

- The <u>point spectrum</u> $\sigma_p(T)$ is the set of $\lambda \in \mathbb{C}$ such that $T - \lambda \, \mathrm{Id}$ is not one-to-one. That is, there exists $f \in D(T)$ such that $(T - \lambda \, \mathrm{Id}) f = 0$, or $Tf = \lambda f$.

- The <u>continuous spectrum</u> $\sigma_c(T)$ is the set of $\lambda \in \mathbb{C}$ such that $T - \lambda \, \mathrm{Id}$ is one-to-one, $\operatorname{ran}(T - \lambda \, \mathrm{Id})$ is dense, but $(T - \lambda \, \mathrm{Id})^{-1}$ is unbounded.

- The <u>residual spectrum</u> $\sigma_r(T)$ is the set of $\lambda \in \mathbb{C}$ such that $T - \lambda \, \mathrm{Id}$ is one-to-one and $\operatorname{ran}(T - \lambda \, \mathrm{Id})$ is not dense.

<u>Theorem 1.</u> If T is self-adjoint, $\sigma_r(T) = \emptyset$, $\sigma_p \cup \sigma_c \subset \mathbb{R}$, and $\sigma_p \cup \sigma_c$ is closed.

We now construct the evolution operator $U_t = e^{-itH}$ for H self-adjoint but possibly unbounded. This is not immediate and we first do it for bounded operators.

Theorem 2

Let A be a bounded operator on $\mathcal{H}$. Then

(a) The sequence of operators $\left(\sum_{n=0}^{N} \frac{(-it)^n}{n!} A^n \right)$ is Cauchy.

The space of bounded operators on $\mathcal{H}$ is Banach, so that the following operator is well-defined for any fixed $t \in \mathbb{R}$:

$$U_t = \sum_{n \geq 0} \frac{(-it)^n}{n!} A^n \doteq e^{-itA}.$$

(b) For any $s, t \in \mathbb{R}$, $U_s U_t = U_{s+t}$.

(c) $\frac{d}{dt} U_t = -i A U_t$

Let us assume in addition that A is self-adjoint. Then

(d) $U_t^* = U_{-t}$.

(e) U_t is unitary.

Proof: For (a), observe that, if $M < N$,

$$\sum_{n=0}^{N} \frac{(-it)^n}{n!} A^n - \sum_{n=0}^{M} \frac{(-it)^n}{n!} A^n = \sum_{n=M+1}^{N} \frac{(-it)^n}{n!} A^n.$$

The norm of the left side is then less than $\sum_{n \geq M+1} \frac{|t|^n}{n!} \|A\|^n$, which is small when M, N are large. This proves (a).

For (b), we have

$$U_s U_t = \sum_{m,n \geq 0} \frac{(-is)^m}{m!} \frac{(-it)^n}{n!} A^m A^n = \sum_{k \geq 0} \frac{1}{k!} A^k \underbrace{\sum_{n=0}^{k} \frac{k!}{(k-n)!\,n!} (-is)^{k-n} (-it)^n}_{(-i(s+t))^k}$$

$$= U_{s+t}.$$

(c) is also easy, since all series converge:

$$\frac{d}{dt} U_t = \lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \left(U_{t+\varepsilon} - U_t \right) = \lim_{\varepsilon \to 0} \sum_{n \geq 0} \frac{(-i(t+\varepsilon))^n - (-it)^n}{n!\,\varepsilon} A^n$$

$$= -i \sum_{n \geq 1} \frac{(-it)^{n-1}}{(n-1)!} A^n = -i A U_t.$$

(d) follows from the Taylor series. And (e) follows from (b) and (d), since $U_t^* U_t = U_{-t} U_t = \mathrm{Id}$, same for $U_t U_t^*$. $\square$

Lemma 3. Let A be a self-adjoint operator. For any $\delta \in \mathbb{R} \setminus \{0\}$,

$$\| (A + i\delta)^{-1} \| \leq \frac{1}{|\delta|}.$$