Proof: $\| (A+i\delta)^{-1} \| = \sup_{\substack{f \in \operatorname{ran}(A+i\delta) \\ f \neq 0}} \frac{\|(A+i\delta)^{-1} f\|}{\|f\|} = \sup_{\substack{g \in D(A) \\ g \neq 0}} \frac{\|g\|}{\|(A+i\delta) g\|}$

$$= \Big(\inf_{\substack{g \in D(A) \\ \|g\|=1}} \| (A+i\delta) g \| \Big)^{-1}.$$

For any $g \in D(A)$, we have

$$\begin{aligned} \| (A+i\delta) g \| &= \big((A+i\delta) g, (A+i\delta) g \big) \\ &= \|Ag\|^2 - i\delta (g, Ag) + i\delta (Ag, g) + \delta^2 \|g\|^2 \\ &\geq \delta^2 \|g\|^2. \end{aligned}$$

□

We now introduce the resolvent operator of H,

$$R_\lambda = (H - i\lambda)^{-1}$$

and Yosida's approximation

$$H_\lambda = -i\lambda H R_\lambda . \qquad (= -i\lambda R_\lambda H)$$

Lemma 4

(a) $\| H R_\lambda \| \leq 2$, so that H_λ is bounded.

(b) For any $f \in D(H)$, $\lim_{\lambda \to \infty} H_\lambda f = Hf$.

Proof: For (a), $\| R_\lambda H f \| = \| R_\lambda (H - i\lambda) f + i\lambda R_\lambda f \| \leq 2 \|f\|$.

We used $\|R_\lambda\| \leq 1/|\lambda|$, from Lemma 3.

For (b), let us first establish that $\lim_{\lambda \to \infty} H R_\lambda f = 0$ for every $f \in \mathcal{H}$. It is enough to prove it for $f \in D(H)$, the general case follows from (a) and a continuity argument. We have

$$\lim_{\lambda \to \infty} \| H R_\lambda f \| = \lim_{\lambda \to \infty} \| R_\lambda (Hf) \| \leq \lim_{\lambda \to \infty} \frac{\|Hf\|}{|\lambda|} = 0 .$$

We used Lemma 3. From $1 = (H - i\lambda) R_\lambda$, we get the identity

$$H R_\lambda = 1 + i\lambda R_\lambda \qquad (*)$$

Then

$$\lim_{\lambda \to \infty} \| H_\lambda f - Hf \| = \lim_{\lambda \to \infty} \| (-i\lambda R_\lambda - 1) Hf \| = \lim_{\lambda \to \infty} \| H R_\lambda H f \| = 0 .$$

□

Since H_λ is bounded, we can define $U_t^{(\lambda)} = e^{-itH_\lambda}$ by Theorem 2. It is bounded, and the ~~bound~~ norm is uniform in λ:

Lemma 5. We have $\| U_t^{(\lambda)} \| \leq 1 \quad \forall \lambda \geq 0, t \geq 0$.

Proof: Using (*) above, we have $H_\lambda = -i\lambda + \lambda^2 R_\lambda$. Then

$$\| U_t^{(\lambda)} \| = \| e^{-t\lambda} e^{-it\lambda^2 R_\lambda} \| \leq e^{-t\lambda} \sum_{n \geq 0} \frac{|t\lambda^2|^n}{n!} \underbrace{\|R_\lambda\|^n}_{\leq \frac{1}{\lambda^n}} \leq 1.$$

□

Theorem 6

As $\lambda \to \infty$, $U_t^{(\lambda)}$ converges strongly to the unitary operator U_t that satisfies $U_s U_t = U_{s+t}$ and $\frac{d}{dt} U_t = -iHU_t = -iU_t H$.

Proof: We first show that $\lim_{\lambda\to\infty} U_t^{(\lambda)} f$ exists for every $f \in \mathcal{H}$.

We have

$$\left\| \frac{\partial}{\partial s} U_{t-s}^{(\lambda)} U_s^{(\mu)} f \right\| = \| U_{t-s}^{(\lambda)} (H_\mu - H_\lambda) U_s^{(\mu)} f \|$$

$$= \| U_{t-s}^{(\lambda)} U_s^{(\mu)} (H_\mu - H_\lambda) f \| \leq \| (H_\mu - H_\lambda) f \| .$$

Since $U_t^{(\lambda)} f - U_t^{(\mu)} f = -\int_0^t \frac{\partial}{\partial s} U_{t-s}^{(\lambda)} U_s^{(\mu)} f \, ds$, we have

$$\| U_t^{(\lambda)} f - U_t^{(\mu)} f \| \leq t \, \| H_\mu f - H_\lambda f \| \to 0$$

as $\mu, \lambda \to \infty$, $\forall f \in D(H)$. A continuity argument shows that $U_t^{(\lambda)} f - U_t^{(\mu)} f \to 0$ $\forall f \in \mathcal{H}$. Then U_t exists, and it is bounded by the uniform boundedness theorem.

The group property is immediate since it holds for $U_t^{(\lambda)}$.

Finally, we have the identity

$$\frac{1}{h} \left(U_{t+h}^{(\lambda)} f - U_t^{(\lambda)} f \right) = -\frac{i}{h} \int_t^{t+h} U_s^{(\lambda)} H f \, ds$$

Taking the limits $\lambda \to \infty$ then $h \to 0$, we get $\frac{d}{dt} U_t f = -iHU_t f$ $\forall f \in D(H)$. $\square$

The counterpart to Theorem 6 is Stone's theorem.

Theorem 7 (Stone)

Let U_t be a strongly continuous one-parameter group of unitary operators on $\mathcal{H}$. (That is, $U_s U_t = U_{s+t}$, and $\lim_{s\to t} \| U_s f - U_t f \| = 0$ $\forall f \in \mathcal{H}$.)

Then there exists a (densely-defined) self-adjoint operator A such that

$$U_t = e^{itA}.$$

We skip the proof.