6. FEYNMAN-KAC FORMULA

6.1 Integral kernels and Perron-Frobenius theorem.

In $\mathbb{C}^n$ an operator is represented by a matrix. Let A be a matrix and (a_{ij}) its matrix elements. Given $f \in \mathbb{C}^n$, we have the relations

$$f(i) = (e_i, f)$$
$$a_{ij} = (e_i, A e_j)$$
$$(Af)(i) = \sum_{j=1}^{n} a_{ij} f(j)$$

where $(e_i)_{i=1}^n$ the chosen orthonormal basis of $\mathbb{C}^n$. There are two natural generalisations of these relations to spaces of functions such as $L^2(\mathbb{R})$.

The first generalisation consists of choosing an orthonormal basis $(e_i)_{i\geq 1}$ (it is countable, since $L^2(\mathbb{R})$ is separable). Then if A is a bounded operator on $L^2(\mathbb{R})$, let

$$a_{ij} = (e_i, A e_j)$$

Then
$$(e_i, Af) = \sum_{j\geq 1} a_{ij} (e_j, f);$$

the sum is now a series, and it is absolutely convergent for all $f \in L^2(\mathbb{R})$ (this follows from Cauchy-Schwarz inequality).

The second generalisation is to replace $f(i)$, $i=1,\dots,n$, by a fct $f(x)$, $x \in \mathbb{R}^d$, that depends on continuous parameters; and by replacing the sum $\sum_j a_{ij} f(j)$ by an integral. That is, let $a(x,y)$ a function on $\mathbb{R}^d \times \mathbb{R}^d$ and define the operator A by

$$(Af)(x) = \int a(x,y) f(y)\, dy .$$

Such operator is called an integral operator and the function $a(x,y)$ is the integral kernel.

Exercise 15. Assume that $a(x,y) \in L^2(\mathbb{R}^d \times \mathbb{R}^d)$. Show that $\int a(x,y) f(y)\, dy$ is absolutely convergent for almost every x, and that $\|A\|^2 \leq \int |a(x,y)|^2 dx\, dy$.
(The latter is the "Hilbert-Schmidt norm". If it is finite, the operator is compact. This is too restrictive for our purpose.)

Exercise 16. Let $R = \sup_{x\in\mathbb{R}^d} \int_{\mathbb{R}^d} |a(x,y)|\, dy$ and $\mathfrak{R} = \sup_y \int |a(x,y)|\, dx$.
(Use the essential supremum if the functions are not continuous.)
Show that
$$\|A\| \leq \sqrt{R\,\mathfrak{R}} .$$

Hint: Use Jensen inequality.

Exercise 17: Show that (i) $a(x,y) = \overline{a(y,x)}$ if A is symmetric; (ii) if $\mathrm{Tr}|A| < \infty$, then $\mathrm{Tr}\, A = \int a(x,x)\, dx$.

Useful information follows from properties of the integral kernel. A relevant example is the Perron-Frobenius theorem. Let us recall the result about matrices: If A is a matrix with positive entries, there is a positive larger eigenvalue (i.e. λ such that all other eigenvalues λ' satisfy $|\lambda'| < \lambda$), with multiplicity 1, and a positive eigenvector (i.e. $v = (v_i)$ with $v_i > 0$ $\forall i$ such that $Av = \lambda v$). And other eigenvectors cannot be all positive.

We consider here a generalisation to integral operators.

Theorem 6.1 (Perron-Frobenius)

Let $a(x,y) > 0$ for almost every $x, y \in \mathbb{R}^d$ and let A be the corresponding integral operator. We assume that $\|A\| < \infty$ and that $\|A\|$ is an eigenvalue. Then $\|A\|$ has multiplicity 1 and its eigenvector ψ can be taken positive, $\psi(x) > 0$ a.e.

Proof: If $\int a(x,y)\, \psi(y)\, dy = \|A\|\, \psi(x)$, then $\mathrm{Re}\, \psi(x)$ (and $\mathrm{Im}\, \psi(x)$) is also eigenvector since $a(x,y) \in \mathbb{R}$; then there exist real eigenvectors, but we need to show that they can be positive. It is clear that

$$(A|\psi|)(x) \geq |A\psi(x)|.$$

Then $$(|\psi|, A|\psi|) \geq (|\psi|, |A\psi|) = \|A\|\, (|\psi|, |\psi|) = \|A\|\, \|\psi\|^2.$$

On the other hand, we have

$$(|\psi|, A|\psi|) \leq \||\psi|\|\, \|A|\psi|\| \leq \|A\|\, \|\psi\|^2.$$

All above inequalities are therefore identities, so that $|\psi| \sim A|\psi|$ and $|\psi|$ is also eigenvector, obviously with eigenvalue $\|A\|$. It follows from

$$|\psi(x)| = \frac{1}{\|A\|} \int a(x,y)\, |\psi(y)|\, dy$$

that $|\psi(x)| > 0$ a.e. The inequalities-identities above also imply that $A|\psi(x)| = |A\psi(x)|$ a.e. so that

$$\int a(x,y)\, |\psi(y)|\, dy = \left| \int a(x,y)\, \psi(y)\, dy \right|.$$

Then $\psi(y) = e^{i\theta} |\psi(y)|$. Thus all eigenvectors with eigenvalue $\|A\|$ can be taken positive and the multiplicity is 1 since it is impossible to find two orthogonal positive functions.

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