6.2. Wiener measure and Feynman-Kac formula

Given an arbitrary self-adjoint operator H on the Hilbert space $\mathcal{H}$, we have seen how to construct the operator e^{-itH}. The latter operator is unitary, it satisfies the group property, and we have $\frac{d}{dt} e^{-itH} = -iH e^{-itH}$. The same method allows to construct the operator e^{-tH} provided H is bounded below.

Theorem 6.2

Assume that H is self-adjoint and that

$$E_0 \doteq \inf_{f \in D(H)} (f, Hf) > -\infty .$$

Then for every $t \geq 0$ there exists an operator e^{-tH} that satisfies

(i) Strong continuity: $\lim_{s \to t} \| e^{-sH} f - e^{-tH} f \| = 0 \;\; \forall f \in \mathcal{H}$.

(ii) Boundedness: $\| e^{-tH} \| \leq e^{-tE_0}$.

(iii) Semigroup property: $e^{-sH} e^{-tH} = e^{-(s+t)H} \;\; \forall s,t \geq 0$

(iv) Derivative: $\frac{d}{dt} e^{-tH} = -H e^{-tH} = -e^{-tH} H$.

Exercise 18. Prove this theorem! You may adapt Thms/Lemmas 2-6 of the mathematical complement.

Next we consider the operator $-\frac{\sigma^2}{2}\Delta$ on $L^2(\mathbb{R}^d)$. It is bounded below, we actually have $-\Delta \geq 0$. So $e^{\frac{t\sigma^2}{2}\Delta}$ exists and is bounded.

Let g_t denote the Gaussian function with mean 0 and variance σ^2, that is,

$$g_t(x) = \frac{1}{(2\pi\sigma^2 t)^{d/2}} e^{-\frac{x^2}{2\sigma^2 t}} .$$

Proposition 6.3

The operator $e^{\frac{t\sigma^2}{2}\Delta}$ is an integral operator with integral kernel $g_t(x-y)$. That is,

$$\left(e^{\frac{t\sigma^2}{2}\Delta} f\right)(x) = \int_{\mathbb{R}^d} g_t(x-y) f(y)\, dy .$$

Proof: We already know that $e^{\frac{t\sigma^2}{2}\Delta}$ and $g_t(x-y)$ are bounded operators, hence continuous. It is enough to check the identity for a dence set of functions. Taking the Fourier transform, the identity becomes

$$\int_{\mathbb{R}^d} dx\, e^{-ikx} \left(e^{\frac{t\sigma^2}{2}\Delta} f\right)(x) = \hat{g}_t(k) \hat{f}(k)$$

with $\hat{g}_t(k) = e^{-\frac{t\sigma^2}{2}k^2}$. Let f be a Schwartz function and fix k. We have

$$\lim_{t\to 0+} \int dx\, e^{-ikx} \left(e^{\frac{t\sigma^2}{2}\Delta} f\right)(x) = \hat{f}(k) = \lim_{t \to 0+} \hat{g}_t(k) \hat{f}(k).$$

Also, $\quad \frac{d}{dt} \hat{g}_t(k) \hat{f}(k) = -\frac{\sigma^2}{2} k^2 \hat{g}_t(k) \hat{f}(k),$

and $\quad \frac{d}{dt} \int dx\, e^{-ikx} \left(e^{\frac{t\sigma^2}{2}\Delta} f\right)(x) = \int dx\, e^{-ikx} \left[\frac{\sigma^2}{2} \Delta\, e^{\frac{t\sigma^2}{2}\Delta} f \right](x) =$

$$\overset{\text{by parts}}{=} -\frac{\sigma^2}{2}k^2 \int dx\, e^{-ikx} \left(e^{t\frac{\sigma^2}{2}\Delta} f\right)(x).$$

For fixed k, as function of t, we have checked that both sides of the equation are equal at $t=0$ and they satisfy the same differential equation. They must be equal for all t. $\square$

We now describe the Wiener measure on $\mathbb{R}^d$. Let C_x be the space of continuous paths $\omega: [0,\infty) \longrightarrow \mathbb{R}^d$ such that $\omega(0) = 0$. Introduce the smallest σ-algebra that contains all sets of the form

$$\{\omega \in C_x : \omega(t_1) \in I_1, \dots, \omega(t_n) \in I_n\}$$

For arbitrary n, arbitrary $0 < t_1 < \dots < t_n$, and arbitrary open sets $I_1, \dots, I_n$ in $\mathbb{R}^d$. Let Σ the σ-algebra.

Consider $n \in \mathbb{N}$, $0 < t_1 < \dots < t_n$, and a function $f: \mathbb{R}^{dn} \to \mathbb{R}$. This defines a function $F: C_x \to \mathbb{R}$ by setting

$$F(\omega) = f(\omega(t_1), \dots, \omega(t_n)).$$

Theorem 6.4

There exists a unique probability measure μ on (C_x, Σ) such that the Lebesgue integral of a function F as above is given by

$$\int_{C_x} F(\omega)\, d\mu(\omega) = \int_{\mathbb{R}^d} dx_1 \dots \int_{\mathbb{R}^d} dx_n\, g_{t_1}(x - x_1)\, g_{t_2-t_1}(x_1 - x_2) \dots g_{t_n - t_{n-1}}(x_{n-1} - x_n)\, f(x_1, \dots, x_n)$$

A useful property of the Wiener measure is that it concentrates on paths that are more than continuous: they are Hölder continuous.

Definition: Let $\alpha \in [0,1]$. A path $\omega : [0,T] \to \mathbb{R}^d$ is Hölder continuous with parameter α if there exists a constant K such that

$$\|\omega(s) - \omega(t)\| < K |s-t|^\alpha$$

for all $s,t \in [0,T]$.

We let $H^\alpha_x([0,T])$ denote the set of paths of C_x that are Hölder continuous with parameter α at each $t \in [0,T]$.

Theorem 6.5

The set $H^\alpha_x([0,T])$ has probability 1 $\forall T > 0$, for all $\alpha < \frac{1}{2}$.
And $H^\alpha_x([0,T])$ has probability 0 if $\alpha \geq \frac{1}{2}$.

A common situation is that we would like to understand $\int F(\omega)\, d\mu(\omega)$ where the function F is the pointwise limit of functions F_n as above. It is enough to check that $F_n(\omega) \to F(\omega)$ for $\omega \in H^\alpha_x$ with any $\alpha < \frac{1}{2}$, rather than for arbitrary continuous ω. In particular, if V is a function $\mathbb{R}^d \to \mathbb{R}$ with sufficient regularity, we can define the integral, for $\omega \in H^\alpha_x$,

$$\int_0^T V(\omega(t))\, dt = \lim_{n\to\infty} \frac{T}{n} \sum_{j=1}^n V\left(\omega\left(j\frac{T}{n}\right)\right).$$