6.2. Wiener measure and Feynman-Kac formula.

Given an arbitrary self-adjoint operator $H$ on the Hilbert space $\mathcal{H}$, we have seen how to construct the operator $e^{-itH}$. The latter operator is unitary, it satisfies the group property, and we have $\frac{d}{dt} e^{-itH} = -iH e^{-itH}$. The same method allows to construct the operator $e^{-tH}$ provided $H$ is bounded below.

Theorem 6.2

Assume that $H$ is self-adjoint and that

$$ E_0 = \inf_{x \in \mathcal{H}} \langle x, Hx \rangle > -\infty. $$

Then for every $T \geq 0$ there exists an operator $e^{-tH}$ that satisfies

(i) Strong continuity: $\lim_{s \to t} \| e^{-sH} - e^{-tH} \| = 0$ for all $t, s \geq 0$.

(ii) Boundedness: $\| e^{-tH} \| \leq e^{-tE_0}$.

(iii) Semigroup property: $e^{-tH} e^{-sH} = e^{-(t+s)H}$ for all $t, s \geq 0$.

(iv) Derivative: $\frac{d}{dt} e^{-tH} = -H e^{-tH} = -e^{-tH} H$.

Exercise 6.2. Prove this theorem! You may adapt Thm/Lemma 2-6 of the mathematical complement.

Next we consider the operator $-\frac{\Delta}{2}$ on $L^2(\mathbb{R}^d)$. It is bounded below, we actually have $-\Delta \geq 0$. So $e^{\frac{\Delta}{2}}$ exist and is bounded.

Let $g_H$ denote the Gaussian function with mean 0 and variance $\sigma^2$, that is,

$$ g_H(x) = \frac{1}{(2\pi \sigma^2)^{d/2}} e^{-\frac{x^2}{2\sigma^2}}. $$

Proposition 6.3

The operator $e^{\frac{\Delta}{2}} f$ is an integral operator with integral kernel $g_H(x-y)$. That is,

$$ (e^{\frac{\Delta}{2}} f)(x) = \int_{\mathbb{R}^d} g_H(x-y) f(y) dy. $$

Proof: We already know that $e^{\frac{\Delta}{2}}$ and $g_H(x-y)$ are bounded operators, hence continuous. It is enough to check the identity for a dense set of functions. Taking the Fourier transform, the identity becomes

$$ \hat{f} = \frac{1}{\sqrt{2\pi}} \frac{\hat{g}_H}{\sigma^2} \hat{f} $$

with $\hat{g}_H(k) = e^{-\frac{|k|^2}{2\sigma^2}}$. Let $f$ be a Schwartz function and pick $\chi$.

We have

$$ \lim_{T \to +\infty} \int_{\mathbb{R}^d} e^{i wk} \frac{1}{\sqrt{2\pi}} \frac{\hat{g}_H}{\sigma^2} \hat{f}(k) = \hat{f}(k) = \lim_{T \to +\infty} \hat{f}_T(k). $$

Also,

$$ \frac{1}{\sqrt{2\pi}} \frac{\hat{g}_H}{\sigma^2} (\hat{f}_T \hat{g}_H)(k) = \hat{f}_T(k) \hat{g}_H(k) $$

and

$$ \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}^d} e^{i wk} \frac{1}{\sqrt{2\pi}} \frac{\hat{g}_H}{\sigma^2} \hat{f}(k) = \int_{\mathbb{R}^d} e^{i wk} \frac{1}{\sqrt{2\pi}} \frac{\hat{g}_H}{\sigma^2} \hat{f}(k) \hat{g}_H(w) dw = \hat{f}_T(k). $$

43
A useful property of the Wiener measure is that it concentrates on paths that are more than continuous: they are Hölder continuous.

**Definition:** Let $x \in [0,1]$. A path $\omega : [0,1] \to \mathbb{R}^d$ is Hölder continuous with parameter $\alpha$ if there exists a constant $K$ such that

$$||\omega(s) - \omega(t)|| < K|s-t|^\alpha$$

for all $s,t \in [0,1]$.

We let $H^\alpha([0,1])$ denote the set of paths of $C_x$ that are Hölder continuous with parameter $\alpha$ at each $x \in [0,1]$.

**Theorem 6.5**

The set $H^\alpha([0,1])$ has probability 1 as $\alpha \to 0$, for all $\alpha \leq \frac{1}{2}$.

And $H^\alpha([0,1])$ has probability 0 if $\alpha > \frac{1}{2}$.

A common situation is that we would like to understand $\int f(x) d\mu(x)$ where the function $f$ is the pointwise limit of functions $f_n$ as above.

It is enough to check that $f_n(w) \to F(w)$ for $w \in H^\alpha$, with any $\alpha \leq \frac{1}{2}$, rather than for arbitrary continuous $w$.

In particular, if $V$ is a function $\mathbb{R}^d \to \mathbb{R}$ with sufficient regularity, we can define the integral, for $w \in H^\alpha$,

$$\int_0^1 V(\omega(t)) dt = \lim_{n \to \infty} \frac{1}{n} \sum_{j=1}^n V(\omega(j/n)).$$