Theorem 6.6 (Feynman–Kac formula)

Let $H = -\frac{\sigma^2}{2}\Delta + V$ where V is continuous and bounded below. Then

$$\left(e^{-tH} f\right)(x) = \int_{C_x} \exp\left\{-\int_0^t V(\omega(s))\,ds\right\} f(\omega(t))\, d\mu(\omega).$$

In order to prove this theorem, we need the Trotter product formula.

Theorem 6.7 (Trotter product formula)

Let A, B be ~~symmetric~~ self-adjoint operators on the Hilbert space $\mathcal{H}$. We assume that A, B are bounded below and that $D(A) \cap D(B)$ is dense in $\mathcal{H}$. Then, for all $f \in \mathcal{H}$, we have

$$e^{-(A+B)} f = \lim_{n\to\infty} \left(e^{-\frac{1}{n}A} e^{-\frac{1}{n}B}\right)^n f.$$

Proof: It is enough to prove the claim for $f \in D(A) \cap D(B)$.

Writing $S_n = e^{-\frac{1}{n}A} e^{-\frac{1}{n}B}$ and $T_n = e^{-\frac{1}{n}(A+B)}$, and using telescopic sum, we have

$$\begin{aligned}
\left(e^{-\frac{1}{n}A} e^{-\frac{1}{n}B}\right)^n - e^{-(A+B)} &= S_n^n - S_n^{n-1} T_n + S_n^{n-1} T_n - S_n^{n-2} T_n^2 + \dots - T_n^n \\
&= \sum_{k=1}^{n} \left(e^{-\frac{1}{n}A} e^{-\frac{1}{n}B}\right)^{n-k} \left[e^{-\frac{1}{n}A} e^{-\frac{1}{n}B} - e^{-\frac{1}{n}(A+B)}\right] e^{-\frac{k-1}{n}(A+B)}.
\end{aligned}$$

Let us first observe that the set $\{e^{-t(A+B)} f : t \in [0,1]\}$ is compact. Indeed, if $(e^{-t_n(A+B)} f)_{n\geq 1}$ is a sequence, there exists a convergent subsequence (t_{n_k}) in $[0,1]$. Then $(e^{-t_{n_k}(A+B)} f)_{k\geq 1}$ converges.

Next, since $\frac{d}{dt} e^{-tT} f \big|_{t=0} = -Tf$ whenever T is symmetric and bounded below, and $f \in D(T)$, then there exists $f_n^{(T)}$ such that $\|f_n^{(T)}\| \to 0$ as $n \to \infty$, and

$$e^{-\frac{1}{n}T} f = f - \frac{1}{n} Tf + \frac{1}{n} f_n^{(T)}.$$

We have

$$e^{-\frac{1}{n}(A+B)} f = f - \frac{1}{n}(A+B) f + \frac{1}{n} f_n^{(A+B)},$$

$$\begin{aligned}
e^{-\frac{1}{n}A} e^{-\frac{1}{n}B} f &= e^{-\frac{1}{n}A}\left(f - \frac{1}{n} Bf + \frac{1}{n} f_n^{(B)}\right) \\
&= e^{-\frac{1}{n}A} f - \frac{1}{n} e^{-\frac{1}{n}A} Bf + \frac{1}{n} e^{-\frac{1}{n}A} f_n^{(B)} \\
&= f - \frac{1}{n} Af + \frac{1}{n} f_n^{(A)} - \frac{1}{n} Bf + \frac{1}{n} h_n + \frac{1}{n} e^{-\frac{1}{n}A} f_n^{(B)}
\end{aligned}$$

where $h_n = (1 - e^{-\frac{1}{n}A}) Bf$ tends to 0. Then

$$\left\| n\left(e^{-\frac{1}{n}A} e^{-\frac{1}{n}B} - e^{-\frac{1}{n}(A+B)}\right) f \right\| \leq \left\| f_n^{(A)} + h_n + e^{-\frac{1}{n}A} f_n^{(B)} - f_n^{(A+B)} \right\| \longrightarrow 0,$$

as $n \to \infty$. It follows from the uniform boundedness theorem that the operator $n\left(e^{-\frac{1}{n}A} e^{-\frac{1}{n}B} - e^{-\frac{1}{n}(A+B)}\right)$ converges to 0 uniformly on compact set. Then

$$\left\| \left[e^{-\frac{1}{n}A} e^{-\frac{1}{n}B} - e^{-\frac{1}{n}(A+B)}\right] e^{-\frac{k-1}{n}(A+B)} f \right\| = o(n)$$

with $o(n)$ independent of k. The claim follows. $\square$

It is a good exercise to re-prove the theorem in the simpler situation where A and B are bounded operators. We now prove the Feynman–Kac formula.

<u>Proof of Theorem 6.6:</u> Without loss of generality, we can assume that $\sigma^2 = 1$ and $V \geq 0$. Let

$$F_n(\omega) = \exp\Big\{ -\frac{t}{n} \sum_{j=1}^{n} V\big(\omega(j\tfrac{t}{n})\big) \Big\}.$$

The limit $n \to \infty$ exists for every $\omega \in H_x^\alpha$:

$$\frac{\limsup F_n(\omega)}{\liminf F_n(\omega)} \leq \lim_{n\to\infty} \exp\Big\{ \frac{t}{n} \sum_{j=1}^{n} \Big(\sup_{(j-1)\frac{t}{n} \leq s \leq j\frac{t}{n}} V(\omega(s)) - \inf_{(j-1)\frac{t}{n} \leq s \leq \frac{t}{n}} V(\omega(s)) \Big) \Big\}$$

$$\leq \lim_{n\to\infty} \exp\Big\{ \sup_{\substack{y,z \in \mathbb{R}^d :\, \|y-x\| \leq Kt^\alpha \\ \|z-x\| \leq Kt^\alpha \\ \|y-z\| \leq \frac{Kt^\alpha}{n^\alpha}}} |V(y) - V(z)| \Big\}.$$

We used the fact that there exists $K < \infty$ such that $\|\omega(s) - \omega(u)\| \leq K|s-u|^\alpha$. Since a continuous function is uniformly continuous on compact sets, the right side above converges to 1. The limsup is equal to the liminf, so F_n converges indeed pointwise as a function $C_x \to \mathbb{R}$.

We also have $F_n(\omega) \leq 1$ $\forall n, \omega$, which allows to use the dominated convergence theorem.

$$\int_{C_x} e^{-\int_0^t V(\omega(s))\,ds} f(\omega(t))\, d\mu(\omega) = \int_{C_x} \lim_n F_n(\omega) f(\omega(t))\, d\mu(\omega)$$

$$\overset{\text{dom. conv.}}{=} \lim_{n\to\infty} \int_{C_x} F_n(\omega) f(\omega(t))\, d\mu(\omega)$$

$$\overset{\text{def. of Wiener meas.}}{=} \lim_{n\to\infty} \int_{\mathbb{R}^d} dx_1 \dots dx_n\, g_{\frac{t}{n}}(x-x_1) \dots g_{\frac{t}{n}}(x_{n-1}-x_n)\, e^{-\frac{t}{n}\sum_{j=1}^{n} V(x_j)} f(x_n)$$

$$= \lim_{n\to\infty} \Big(\underbrace{e^{\frac{t}{2n}\Delta} e^{-\frac{t}{n}V} \dots e^{\frac{t}{2n}\Delta} e^{-\frac{t}{n}V}}_{n \text{ times}} f \Big)(x)$$

$$\overset{\text{Trotter}}{=} \big(e^{-t(-\frac{1}{2}\Delta + V)} f \big)(x).$$

We also used Proposition 6.3, that claims that $e^{\frac{t}{2}\Delta}$ is an integral operator with integral kernel $g_t(x-y)$. $\square$

Using the same ideas, we can express the integral kernel of the operator $e^{-t(-\frac{1}{2}\Delta+V)}$ using the Wiener measure for Brownian bridges. Let $C_{xy}^{(t)}$ be the space of continuous paths $\omega : [0,t] \to \mathbb{R}^d$ such that $\omega(0) = x$ and $\omega(t) = y$. We equip $C_{xy}^{(t)}$ with the σ-algebra generated by the sets

$$\{\omega \in C_{xy}^{(t)} : \omega(t_1) \in I_1, \dots, \omega(t_n) \in I_n\}$$

for arbitrary n, $0 < t_1 < \dots < t_n < t$, and arbitrary open sets $I_1, \dots, I_n$ in $\mathbb{R}^d$. As before, we consider functions of the form