\[ F(w) = F(w(1), \ldots, w(n)), \]

for some given \( n, h_1, \ldots, h_n \in [0,1], \) and a function \( F: \mathbb{R}^n \rightarrow \mathbb{R}. \)

**Theorem 6.8 (Measure on Brownian bridges)**

There exists a unique measure \( \mu \) on \( C_{0}^{0} \) such that the Lebesgue integral of a function \( F \) as above is given by

\[
\int_{C_{0}^{0}} F(w) \, d\mu(w) = \int_{\mathbb{R}^{n}} \cdots \int_{\mathbb{R}^{n}} g_{1}(x_{1}) g_{2}(x_{2}) \cdots g_{n}(x_{n}) F(x_{1}, \ldots, x_{n}).
\]

This theorem is very similar to Theorem 6.4. Notice that the measure \( \mu \) above is not a probability measure, because \( \mu(C_{0}^{0}) = g_{1}(0) \) (check it!). We also omit the proof of Theorem 6.8, which is usually done using the Rice–Markov Theorem or the Kolmogorov extension theorem.

The measure on Brownian bridges allows to obtain an expression for the integral kernel of the exponential of Schrödinger operators.

**Theorem 6.9 (Feynman–Kac formula for integral kernel)**

Let \( k(x,y) \) denote the integral kernel of \( e^{-tH} \), where \( H = -\frac{1}{2} \Delta \frac{1}{m}V \) with \( V \) continuous and bounded below. Then

\[ k(x,y) = \int_{\mathbb{R}^{n}} \exp \left\{ -\int_{0}^{1} \frac{1}{2} V(w(s)) \, ds \right\} \, d\mu(w). \]

**Proof:** We need to show that \( (e^{-tH}f)(x) = \int k(x,y) f(y) \, dy \), where \( k \) is the kernel above, and for all \( f \) in a dense set. By dominated convergence and Theorem 6.8, we have

\[
\int k(x,y) f(y) \, dy = \lim_{\varepsilon \to 0} \int k(x,y) g_{\varepsilon}(y) \, dy.
\]

Let us assume that \( f \) is continuous, so we can replace \( f(y) \) by \( f(x) \).

Since \( \int g_{\varepsilon}(x-y) \, dy = 1 \), we get the same expression as in the 3rd page of 56, which is equal to \( (e^{-tH}f)(x) \). \( \Box \)

An important consequence of the Feynman–Kac Formula is that the integral kernel of \( e^{-tH} \), where \( H \) is a Schrödinger operator, is positive.

Theorems 6.6 and 6.8 do not apply to the operator \( H = -\frac{1}{2} \Delta - \frac{1}{m}V \) on \( L^{2}(\mathbb{R}^{n}) \) since \( V \) is not bounded below. But we can use a limiting argument. Let

\[ V_{n} = \begin{cases} 
-\frac{1}{n^{2}} & \text{if } \|x\| > \frac{1}{n} \\
-\infty & \text{otherwise},
\end{cases}
\]

so that \( V_{n} \) is continuous and bounded below, and let \( H_{n} = -\frac{1}{2} \Delta + V_{n} \).

From Theorem 6.9, the integral kernel of \( e^{-tH_{n}} \) is

\[ k_{n}(x,y) = \int_{\mathbb{R}^{n}} \exp \left\{ -\int_{0}^{1} \frac{1}{2n} V_{n}(w(s)) \, ds \right\} \, d\mu(w). \]
Proposition 6.10
For every \( f \in L^2(\mathbb{R}^3) \), we have
\[
\left( e^{-tH} \right) \ast f(x) = \int \! k(x,y) f(y) \, dy \quad \text{a.e.}
\]

Proof. First, observe that \( H_n f(x) = H f(x) + \left( \frac{\lambda_n}{\hbar^2} - \frac{\lambda_n}{\hbar^2} \right) \mathbf{1}_{\text{box}} f(x) \).
Then \( H_n f \to H f \) in \( \text{D}(X(H)) \). We accept without proof that 
\[
e^{-tH} f \to e^{-tH} f \quad \text{as } n \to \infty,
\]
for every \( f \in L^2(\mathbb{R}^3) \). Then
\[
\left( e^{-tH} \right) \ast f(x) = \lim_{n \to \infty} \left( e^{-tH_n} \right) \ast f(x) = \lim_{n \to \infty} \int \! k_n(x,y) f(y) \, dy = \int \! k(x,y) f(y) \, dy,
\]
The latter identity holds for every positive or negative \( f \) by monotone convergence. Then it holds for finite sums of such functions, and a density argument extends it to all \( f \in L^2(\mathbb{R}^3) \).

6.3. Application: ground state of the hydrogen atom

Let \( H = -\Delta - \frac{1}{r} \) be the Hamiltonian of the hydrogen atom (we drop \( \hbar \) and \( i \) before the Laplacian). The spectrum and the eigenstates of \( H \) were described before, but there was no proof.
We now show that \(-\frac{1}{4}\) is the bottom of the spectrum.

Proposition 6.11
We have \(-\frac{1}{4} = \inf \sigma(H) \) and \( e^{\frac{1}{4}} = \sup \sigma(e^{-\frac{1}{4}H}) = \| e^{-\frac{1}{4}H} \| \).

We are helped by knowing that \( F_{\infty} = e^{-\frac{1}{4}H} \) is eigenstate of \( H \) with eigenvalue \(-\frac{1}{4} \): Using spherical coordinates,
\[
H F = \left( -\frac{\partial^2}{\partial r^2} - \frac{2}{r} \frac{\partial}{\partial r} - \frac{1}{r^2} \Delta_{\text{Sph}} - \frac{1}{r} \right) e^{-\frac{1}{2}r}
\]
\[
= \left( -\frac{1}{4} + \frac{1}{4} - 0 - \frac{1}{4} \right) e^{-\frac{1}{2}r}
\]
\[
= -\frac{1}{4} F. \]

It then follows that \( F \) is eigenstate of \( e^{-\frac{1}{4}H} \) with eigenvalue \(-\frac{1}{4} \).

Proof. Let \( \lambda_0 = \sup \sigma(e^{-\frac{1}{4}H}) \). Since \( e^{-\frac{1}{4}H} \geq 0 \), \( \lambda_0 = \| e^{-\frac{1}{4}H} \| \).
We know that \( \sigma(e^{-\frac{1}{4}H}) \) is closed and that it consists of point and continuous spectrum. We first check that \( \sup \sigma(e^{-\frac{1}{4}H}) \leq 1 \).