$$F(\omega) = f(\omega(t_1), \ldots, \omega(t_n)),$$

for some given n, $t_1, \ldots, t_n \in [0,t]$, and a function $f : \mathbb{R}^{dn} \to \mathbb{R}$.

<u>Theorem 6.8</u> (Measure on Brownian bridges)

There exists a unique measure μ on $C^{(t)}_{xy}$ such that the Lebesgue integral of a function F as above is given by

$$\int_{C^{(t)}_{xy}} F(\omega)\, d\mu(\omega) = \int_{\mathbb{R}^d} dx_1 \ldots \int_{\mathbb{R}^d} dx_n \; g_{t_1}(x-x_1)\, g_{t_2-t_1}(x_1-x_2) \ldots g_{t-t_n}(x_n-y)\, f(x_1, \ldots, x_n).$$

This theorem is very similar to Theorem 6.4. Notice that the measure μ above is not a probability measure, because $\mu(C^{(t)}_{xy}) = g_t(x-y)$ (check it!). We also omit the proof of Theorem 6.8, which is usually done using the Riesz-Markov theorem or the Kolmogorov extension theorem.

The measure on Brownian bridges allows to obtain an expression for the integral kernel of the exponential of Schrödinger operators.

<u>Theorem 6.9</u> (Feynman-Kac formula for integral kernel)

Let $k(x,y)$ denote the integral kernel of e^{-tH}, where $H = -\frac{\sigma^2}{2}\Delta + V$ with V continuous and bounded below. Then

$$k(x,y) = \int_{C^{(t)}_{xy}} \exp\left\{ -\int_0^t V(\omega(s))\, ds \right\} d\mu(\omega).$$

<u>Proof</u>: We need to show that $(e^{-tH} f)(x) = \int k(x,y) f(y)\, dy$, where k is the kernel above, and for all f in a dense set. By dominated convergence and Theorem 6.8, we have

$$\int k(x,y) f(y)\, dy = \lim_{n\to\infty} \int dy\, f(y) \int dx_1 \ldots dx_n\, g_{\frac{t}{n}}(x-x_1) \ldots g_{\frac{t}{n}}(x_n-y)\, e^{-\frac{t}{n}\sum_{j=1}^{n} V(x_j)}$$

Let us assume that f is continuous, so we can replace $f(y)$ by $f(x_n)$. Since $\int g_{\frac{t}{n}}(x_n-y)\, dy = 1$, we get the same expression as in line 3, page 56, which is equal to $(e^{-tH} f)(x)$. $\square$

An important consequence of the Feynman-Kac Formula is that the integral kernel of e^{-tH}, where H is a Schrödinger operator, is positive.

Theorems 6.6 and 6.8 do not apply to the operator $H = -\frac{1}{2}\Delta - \frac{1}{\|x\|}$ on $L^2(\mathbb{R}^3)$ since $V = -\frac{1}{\|x\|}$ is not bounded below. But we can use a limiting argument. Let

$$V_n = \begin{cases} -\frac{1}{\|x\|} & \text{if } \|x\| > \frac{1}{n} \\ -n & \text{otherwise,} \end{cases}$$

so that V_n is continuous and bounded below, and let $H_n = -\frac{1}{2}\Delta + V_n$. From Theorem 6.9, the integral kernel of e^{-tH_n} is

$$k_n(x,y) = \int_{C^{(t)}_{xy}} \exp\left\{ -\int_0^t V_n(\omega(s))\, ds \right\} d\mu(\omega).$$

The limit of $e^{-\int_0^t V_n(\omega(s))ds}$ exists, in the sense that it belongs to $[0,\infty]$. Then $k(x,y) = \lim_{n\to\infty} k_n(x,y)$ exists by the monotone convergence theorem. What remains to be checked is that $k(x,y)$ is the integral kernel of e^{-tH}.

Proposition 6.10

For every $f \in L^2(\mathbb{R}^3)$, we have

$$\left(e^{-tH} f\right)(x) = \int k(x,y) f(y)\, dy \quad \text{a.e.}$$

Proof: First, observe that $H_n f(x) = Hf(x) + \left(\frac{1}{\|x\|} - \frac{1}{n}\right) 1_{\|x\|<\frac{1}{n}} f(x)$.

Then $H_n f \to Hf$ $\forall f \in \mathcal{D}(H)$. We accept without proof that

$$e^{-tH_n} f \longrightarrow e^{-tH} f \quad \text{as } n\to\infty,$$

for every $t \geq 0$ and $f \in L^2(\mathbb{R}^3)$. Then

$$\left(e^{-tH} f\right)(x) = \lim_{n\to\infty} \left(e^{-tH_n} f\right)(x) = \lim_{n\to\infty} \int k_n(x,y) f(y)\, dy = \int k(x,y) f(y)\, dy.$$

The latter identity holds for every positive or negative f by monotone convergence. Then it holds for finite sums of such functions, and a density argument extends it to all $f \in L^2(\mathbb{R}^3)$. $\square$

6.3. Application: ground state of the hydrogen atom

Let $H = -\Delta - \frac{1}{\|x\|}$ be the Hamiltonian of the hydrogen atom (we drop the $\frac{1}{2}$ before the Laplacian). The spectrum and the eigenstates of H were described before, but there were no proofs. We now show that $-\frac{1}{4}$ is the bottom of the spectrum:

Proposition 6.11

We have $-\frac{1}{4} = \inf \sigma(H)$ and $e^{\frac{1}{4}} = \sup \sigma(e^{-H}) = \|e^{-H}\|$.

We are helped by knowing that $f(x) = e^{-\frac{1}{2}\|x\|}$ is eigenstate of H with eigenvalue $-\frac{1}{4}$: Using spherical coordinates,

$$Hf = \left(-\frac{\partial^2}{\partial r^2} - \frac{2}{r}\frac{\partial}{\partial r} - \frac{1}{r^2}\Delta_{sph} - \frac{1}{r}\right) e^{-\frac{1}{2}r}$$

$$= \left(-\frac{1}{4} + \frac{1}{r} - 0 - \frac{1}{r}\right) e^{-\frac{1}{2}r}$$

$$= -\frac{1}{4} f.$$

It then follows that f is eigenstate of e^{-H} with eigenvalue $e^{\frac{1}{4}}$.

Proof: Let $\lambda_0 = \sup \sigma(e^{-H})$. Since $e^{-H} \geq 0$, $\lambda_0 = \|e^{-H}\|$. We know that $\sigma(e^{-H})$ is closed and that it consists of point and continuous spectrum. We first check that $\sup \sigma_c(e^{-H}) \leq 1$.