If $2 \in \sigma_c(e^H)$, there exists a Weyl sequence $(s_n)$ such that $\|e^{-\frac{H}{n}} s_n\| \to 0$. Then $\|e^{-\frac{H}{n}} g\| \to 0$. But
\[
(e^{-\frac{H}{n}} g_n, e^{-\frac{H}{n}} g_n) = \int \left( \int dy \, k(x, y) g_n(x) g_n(y) \right) dx
\]
\[
= \int \left( \int dy \, k(x, y) g(x) g(y) \right) dx + \int \left( \int dy \, k(x, y) g_n(x) g(y) \right) dx
\]
\[
\to 0 \text{ as } n \to \infty.
\]
For that large, $(k_n)$ is ippoincised.

\[
\leq C_n(R) + \epsilon \left( e^{-\lambda_{j+1}}(x), e^{-\lambda_j}(x) \right)
\]
where $C_n(R) \to 0$ as $n \to \infty$, and $c_n \to 0$ as $n \to \infty$.

Then $2^2 \leq 1$, so $\sigma_c(e^H) \subset [0, 1]$.

So we know that $2 \in \sigma_c(e^H)$, thus $2 \in \sigma_c(e^H)$.

By Theorem 6.1 (Poincaré–Frobenius), there exist a non-negative eigenvalue $\lambda_0$. Then $(\lambda_0) \neq 0$.

Further,
\[
e^{-\lambda_0} (f, g) = (e^{\lambda_0} f, e^{-\lambda_0} g) = (f, e^{-\lambda_0} g) = 2 \lambda_0 (f, g).
\]

Then $\lambda_0 = e^{-\lambda_0}$.

Proposition 6.11 may seem to be rather specialised. But we can turn it into a nice operator inequality.

Corollary 6.12. In $L^2(R^n)$, we have
\[
-\Delta \geq \frac{1}{4 \pi^2} - \frac{1}{4},
\]
with equality if and only if $e^{-\frac{H}{n}} g \in L^2(R^n)$.

Recall the inequality of Proposition 5.1: $-\Delta \geq \frac{1}{4 \pi^2}$ on $L^2(R^n)$.

Let us compare the plots of these functions:

Proof: Since $-\frac{1}{4} = \inf \sigma(H)$, we have $(f, (-\Delta - \frac{1}{4} + \frac{i}{2}) f) \geq 0$, for all $f \in D(-\Delta)$. That is, $(f, (-\Delta - \frac{1}{4} + \frac{i}{2}) f) \geq 0$, for all $f \in D(-\Delta)$. \qed
7. CONCLUSION

This was a necessarily brief introduction to quantum mechanics. We have only touched a few aspects. But it should now be easier to learn further topics.

Let us conclude by listing several topics of interest:

1. The relation between self-adjoint operators and physical observables is still a controversial subject. Bell inequalities.

2. A system of $N$ particles, each with state space $\mathbb{F}_i$, has state space $\mathcal{H} = \bigotimes_{i=1}^N \mathbb{F}_i$.

3. Bosons & Fermions: The state space for $N$ identical bosons, resp. $N$ identical fermions, is $\mathcal{H}_N^{(b)} = \mathbb{F}_1 \otimes \cdots \otimes \mathbb{F}_N$, resp.

4. The Fock space for an arbitrary number of particles is the Hilbert space $\mathcal{F} = \bigoplus_{N=0}^{\infty} \mathcal{H}_N^{(b)}$.

5. Spin operators, their relation to the angular momentum, and their representation on $\mathbb{C}^n$.


7. Lattice systems, that are relevant to the study of the electronic properties of condensed matter. They are easier to define (Hilbert space are often finite), but they are very difficult to study.

8. Quantum information theory.