If $\lambda \in \sigma_c(e^{-H})$, there exists a Weyl sequence (g_n) such that $\|(e^{-H}-\lambda)g_n\| \to 0$. Then $\|e^{-H}g_n\| \to \lambda$. But

$$(e^{-H}g_n, e^{-H}g_n) = \int dx \left|\int dy\, k(x,y)\, g_n(y)\right|^2$$

$$= \underbrace{\int_{\|x\|<R} dx \Big|\underbrace{\int dy\, k(x,y)\, g_n(y)}_{\to 0\ \forall x,\ \text{because } g_n \text{ is Weyl}}\Big|^2}_{\to 0 \text{ by dominated convergence}} + \int_{\|x\|>R} dx \left|\int dy\, k(x,y)\, g_n(y)\right|^2$$

for $\|x\|$ large, $k(x,y) \approx$ gaussian kernel

$$\leq C_n(R) + e^{c_R} \underbrace{\left(e^{\Delta}|g_n|, e^{\Delta}|g_n|\right)}_{\leq 1}$$

where $C_n(R) \to 0$ as $n \to \infty$, and $c_R \to 0$ as $R \to \infty$.
Then $\lambda^2 \leq 1$, so $\sigma_c(e^{-H}) \subset [0,1]$.

So we know that $\lambda_0 \notin \sigma_c(e^{-H})$, thus $\lambda_0 \in \sigma_p(e^{-H})$.
By Theorem 6.1 (Perron – Frobenius), there exists a non-negative eigenvector g. Then $(f,g) \neq 0$. Further,

$$e^{\frac{1}{4}}(f,g) = (e^{-H}f, g) = (f, e^{-H}g) = \lambda_0 (f,g).$$

Then $\lambda_0 = e^{\frac{1}{4}}$. □

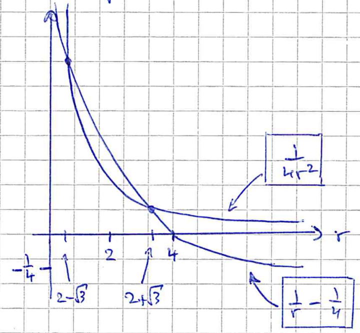
Proposition 6.11 may seem to be rather specialised. But we can turn it into a nice operator inequality.

<u>Corollary 6.12.</u> In $L^2(\mathbb{R}^3)$, we have

$$-\Delta \geq \frac{1}{\|x\|} - \frac{1}{4},$$

with equality when applied to the function $e^{-\frac{1}{2}\|x\|}$.

Recall the inequality of Proposition 5.1: $-\Delta \geq \frac{1}{4\|x\|^2}$ on $L^2(\mathbb{R}^3)$.
Let us compare the plots of these functions:



<u>Proof:</u> Since $-\frac{1}{4} = \inf \sigma(H)$, we have $(f, (-\Delta - \frac{1}{\|x\|})f) \geq -\frac{1}{4}$, that is, $(f, (-\Delta - \frac{1}{\|x\|} + \frac{1}{4})f) \geq 0$, $\forall f \in$ dense set. □

7. CONCLUSION

This was a necessarily brief introduction to quantum mechanics. We have only touched a few aspects. But it should now be easier to learn further topics.

Let us conclude by listing several topics of interest.

1. The relations between self-adjoint operators and physical observables is still a controversial subject. Bell inequalities.

2. A system of N particles, each with state space h_i, has state space
$$\mathcal{H} = \bigotimes_{i=1}^{N} h_i$$

3. Bosons & Fermions: The state space for N identical bosons, resp. N identical fermions, is $\mathcal{H}_N^{(+)} = P_{sym}\, h^{\otimes N}$, resp. $\mathcal{H}^{(-)} = P_{antisym.}\, h^{\otimes N}$. Bose-Einstein condensation; Pauli exclusion principle.

4. The Fock space for an arbitrary number of particles is the Hilbert space
$$\mathcal{F}^{(\pm)} = \bigoplus_{N \geq 0} \mathcal{H}_N^{(\pm)}$$

5. Spin operators, their relations to the angular momentum, and their representation on $\mathbb{C}^n$.

6. Systems with (massive) quantum particles and quantised radiation field (photons).

7. Lattice systems, that are relevant to the study of the electronic properties of condensed matter. They are easier to define (Hilbert spaces are often finite), but they are very difficult to study.

8. Quantum information theory.