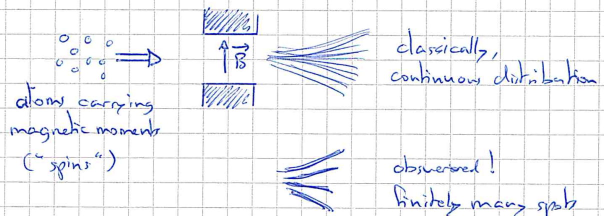
There are further intriguing facts, such as the spectra of light emission and absorbtion of atoms. The hydrogen is characterised by Balmer's formula:

$$\nu = \frac{me^4}{2\hbar^2}\left(\frac{1}{n^2} - \frac{1}{m^2}\right), \quad n, m \in \mathbb{N}, \quad m > n.$$

Bohr suggested that atoms have "energy levels", $E_n = -\frac{me^4}{2\hbar^2}\frac{1}{n^2}$, $n \in \mathbb{N}$, for hydrogen.

1922: The experiment of Stern and Gerlach, which points to another quantisation of a quantity that is continuous classically.

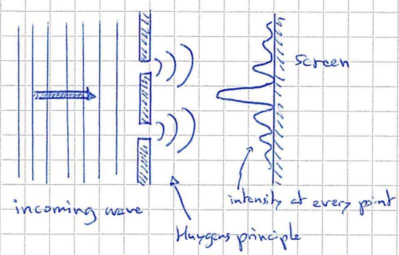


The years 1910–1925 correspond to the "Old quantum theory", with the Bohr–Sommerfeld quantisation rules. These are ad hoc hypotheses, that are made in order to fit experimental data.

1923: de Broglie suggests that matter has wave-like behaviour too (this should be contrasted to Einstein: light has corpuscular behaviour too).

This can be summarised by the existence of a "wave-corpuscule duality".

1927: Experiments on the diffraction of electrons (Davisson & Germer)



Even when electrons are sent one by one, their evolution is wave-like.

From 1923 to 1927, Heisenberg and Schrödinger formulate the basic setting of quantum mechanics. Dirac also made major contributions. Later, the mathematical framework was much enhanced by von Neumann. The influence of quantum mechanics on the development of mathematics is huge, this was the main motivation for functional analysis.

2. SCHRÖDINGER EQUATION

Quantum mechanics describes diverse physical systems, with one or several particles, with or without radiation, with or without spin degrees of freedom, ... It is sometimes necessary to consider simplified models, especially in view of the study of condensed matter systems. In all cases, the state space has the structure of a Hilbert space, the state of the system is given by a vector of norm 1, and the evolution is given by a Schrödinger equation.

Here we start with the explicit description of a single quantum particle in an external potential. The particle is assumed to be "massive", that is, it has a mass. Typical examples are (small) atoms, electrons, etc..., but not photons.

2.1. Setting for a massive particle in external potential.

We first describe the axioms without much motivation. We shall explore some consequences, including the relation to classical mechanics. We shall see later a semi-derivation that is based on natural principles, which shows that there are physical motivations for the setting.

1st axiom: The state space for a quantum particle at given time is $L^2(\mathbb{R}^d, \mathbb{C})$. Given $\Psi \in L^2(\mathbb{R}^d)$, the probability density for observing the particle at $x \in \mathbb{R}^d$ is $|\Psi(x)|^2$.

The function Ψ is called "wave function".

2nd axiom: If $\Psi(x,t)$ describes the evolution of the quantum particle, then it satisfies Schrödinger equation:

$$i\hbar \frac{\partial}{\partial t} \Psi(x,t) = -\frac{\hbar^2}{2m} \Delta \Psi(x,t) + V(x)\, \Psi(x,t),$$

where $\hbar = \frac{h}{2\pi}$, m is the mass, $\Delta = \sum_{i=1}^{d} \frac{\partial^2}{\partial x_i^2}$ is the Laplace operator, and $V: \mathbb{R}^d \to \mathbb{R}$ is the external potential.

A particle also has a velocity, or equivalently a momentum. It is given by the Fourier transform:

$$\hat{\Psi}(k) = \int_{\mathbb{R}^d} e^{-ikx}\, \Psi(x)\, dx, \qquad k \in \mathbb{R}^d.$$

3rd axiom: The probability density for observing the particle with momentum $k \in \mathbb{R}^d$ is $\frac{1}{(2\pi\hbar)^d} \left| \hat{\Psi}\left(\frac{k}{\hbar}, t\right) \right|^2$.

We need $\|\Psi\| = 1$ in order to have probability densities. Recall Plancherel identity, $\|\Psi\| = \frac{1}{(2\pi)^{d/2}} \|\hat{\Psi}\|$.