Here is the second uncertainty principle, which is even simpler to prove. But it only applies to $d=1$.

Theorem 2.5 For every continuous $f \in L^2(\mathbb{R})$ with $\|f\|=1$, we have

$$\sup_{x\in\mathbb{R}} |f(x)|^2 \leq \left(\frac{1}{2\pi} \int_{-\infty}^{\infty} k^2 |\hat{f}(k)|^2 \, dk \right)^{1/2}.$$

Proof: By the inverse Fourier transform, we have for any a,

$$f(x) = \frac{1}{2\pi} \int \frac{dk}{\sqrt{k^2+a^2}} \sqrt{k^2+a^2} \, e^{ikx} \hat{f}(k).$$

Then, using Cauchy-Schwarz inequality,

$$|f(x)|^2 \leq \frac{1}{4\pi^2} \underbrace{\left(\int_{-\infty}^{\infty} \frac{dk}{k^2+a^2} \right)}_{= \frac{1}{a} \arctan \frac{k}{a} \Big|_{-\infty}^{\infty} = \frac{\pi}{|a|}} \left(\int_{-\infty}^{\infty} (k^2+a^2) |\hat{f}(k)|^2 \, dk \right)$$

$$= \frac{1}{4\pi |a|} \int_{-\infty}^{\infty} k^2 |\hat{f}(k)|^2 \, dk + \frac{|a|}{2}.$$

The optimal choice is $a^2 = \frac{1}{2\pi} \int k^2 |\hat{f}(k)|^2 dk$, and the result follows. $\square$

The energy of a particle in the state $f \in L^2(\mathbb{R})$ is

$$\frac{1}{2m} \langle P^2 \rangle + \langle V(X) \rangle = \int_{-\infty}^{\infty} \overline{f(x)} \left[-\frac{\hbar^2}{2m} \Delta f(x) + V(x) f(x) \right] dx$$

The minimal energy for a classical particle in a potential V is $\inf V(x)$. For a quantum particle, it is higher.

Exercise 4: Give an example of a potential $V : \mathbb{R} \to \mathbb{R}$ such that $\inf_x V(x) = -\infty$, but such that the quantum energy is bounded below.

Hint: Use Theorem 2.5, and get a lower bound that involves $\int_{-\infty}^{\infty} |V(x)| \, dx$ rather than $\inf V(x)$.

This exercise is related to the question of stability of matter: why is the energy of electrons orbiting a nucleus bounded below? Otherwise, we could pump arbitrarily much energy out of a single atom!

3. EVOLUTION OPERATOR

3.1 A general solution to the Schrödinger equation

We can reformulate the Schrödinger equation in the language of functional analysis, rather than PDEs. We consider the Hilbert space $\mathcal{H} = L^2(\mathbb{R}^d)$ and the Schrödinger operator

$$H = -\Delta + V$$

where $\Delta = \sum_{i=1}^{d} \frac{\partial^2}{\partial x_i^2}$ and V is the multiplication operator $(Vf)(x) = V(x) f(x)$. We choose the domain

$$D(H) = \{ f \in H^2(\mathbb{R}^d) : Vf \in L^2(\mathbb{R}^d) \}$$

It is clear that $D(H)$ is dense in $L^2(\mathbb{R}^d)$; H is self-adjoint. The evolution of the quantum particle is given by a function $\psi : [0, \infty) \longrightarrow L^2(\mathbb{R}^d)$ that satisfies the equation

$$i \frac{d}{dt} \psi(t) = H \psi(t),$$

with given initial condition $\psi(0)$. The "classical solutions" of Definition 2.1 are solutions. But we can use Theorem 6 of the Math. Complement and write a much more general solution:

$$\psi(t) = e^{-itH} \psi(0).$$

The initial condition can be quite irregular, $\psi(0)$ does not need to be in $D(H)$. If $\psi(t) \in D(H)$, then $i \frac{d}{dt} \psi(t) = H \psi(t)$. It is now obvious that $\| \psi(t) \| = \| \psi(0) \|$, since e^{-itH} is unitary.

3.2 Observables

The axioms of Quantum Mechanics were presented in Chapter 2. They looked rather odd, and they seemed to work by accident. The goal of this section is to present a semi-derivation of the framework of QM, which shows that the axioms were essentially the only possible ones.

Let us admit that the state of the particle is represented by the function $\psi(t) \in L^2(\mathbb{R}^d)$ with $\|\psi(t)\| = 1$. To each "observable" corresponds a self-adjoint operator on $L^2(\mathbb{R}^d)$. The mean-value of the observable A if the particle is in the state $\psi(t)$ is

$$\langle A \rangle_{\psi(t)} = (\psi(t), A \psi(t)).$$

This assumes that $\psi(t) \in D(A)$. The position operator X is the multiplication operator

$$(X f)(x) = x f(x),$$

with domain $D(X) = \{ f \in L^2(\mathbb{R}^d) : x f \in L^2 \}$.