We start with $d=1$. The eigenvalue equation is

$$-\frac{d^2}{dx^2}\psi(x) = \lambda\psi(x), \qquad \psi(0) = \psi(L) = 0.$$

The solutions are $\psi_n(x) = \sin\frac{\pi n x}{L}$, $\lambda_n = \frac{\pi^2 n^2}{L^2}$, $n \in \mathbb{N}$.
They form an orthogonal basis of $L^2([0,L])$. Consequently, there are no other solutions, and the spectrum of H is pure point, with $\sigma_p(H) = \left\{ \frac{\pi^2 n^2}{L^2} : n \in \mathbb{N} \right\}$.

The solution of the problem for arbitrary d follows. The solutions are

$$\psi_{n_1,\dots,n_d}(x) = \prod_{j=1}^{d} \sin\frac{\pi n_j x_j}{L},$$

$$\lambda_{n_1,\dots,n_d} = \frac{\pi^2}{L^2}\sum_{j=1}^{d} n_j^2,$$

for all $n_1,\dots,n_d \in \mathbb{N}$. Since $\bigotimes_{j=1}^{d} L^2([0,L]) \simeq L^2([0,L]^d)$, these functions form an orthogonal basis.

Another relevant potential is the finite square well, where

$$V(x) = \begin{cases} -V_0 & \text{if } x_j \in [0,L] \ \forall j = 1,\dots,d \\ 0 & \text{otherwise.} \end{cases}$$

One can check that $\sigma_c(H) = [0,\infty)$ and $\sigma_p(H) \subset (-V_0, 0)$. See most textbooks, e.g. Gustafson & Sigal.

4.2 The harmonic oscillator

We again start with $d=1$. The Hilbert space is $L^2(\mathbb{R})$ and the Hamiltonian

$$H = -\Delta + x^2.$$

We consider first a few general claims that allow to understand the properties of this class of Hamiltonians. The continuous spectrum of a self-adjoint operator can be characterised by the existence of a Weyl sequence.

<u>Definition 4.1</u>. Let A be a self-adjoint operator on $\mathcal{H}$ and λ be a complex number. A <u>Weyl sequence</u> for (A,λ) is a sequence (f_n) in $D(A)$ such that

(i) $\|f_n\| = 1 \quad \forall n$.

(ii) $\|(A-\lambda)f_n\| \to 0$ as $n \to \infty$ (so f_n is almost an eigenvector).

(iii) $f_n \rightharpoonup 0$ as $n \to \infty$. That is, $(f_n, g) \to 0 \ \forall g \in \mathcal{H}$. Thus (f_n) does not converge to an eigenvector.

<u>Theorem 4.1</u> (Weyl). Let A be self-adjoint on $\mathcal{H}$. Then $\lambda \in \sigma_c(A)$ iff there exists a Weyl sequence.

We skip the proof. We also accept the next result without proof.

Theorem 4.2. Let $V(x)$ be a continuous function on $\mathbb{R}^d$ such that $V(x) \to \infty$ as $\|x\| \to \infty$. Then the spectrum of $H = -\Delta + V$ consists of isolated eigenvalues $\lambda_1 < \lambda_2 < \dots$ such that $\lambda_n \to \infty$ as $n \to \infty$.

Although we do not prove it, we can reflect on why the theorem is true. We know that $\sigma(H) = \sigma_p(H) \cup \sigma_c(H)$. If $\lambda \in \sigma_c(H)$, there exists a Weyl sequence (f_n), which converges weakly to 0. Then f_n must "converge to a Dirac", or have faster and faster oscillations, or disappear at infinity. (Or a combination of the three.) In the first two cases, $\|\Delta f_n\| \to \infty$; in the third cases, $\|V f_n\| \to \infty$. It is impossible for $\|(-\Delta + V - \lambda) f_n\|$ to stay bounded, let alone to vanish.
The proof that $\lambda_n \to \infty$ follows from the minimax theorem.

We have eluded the mention of the domain of H. If V is a "nice" function, such as continuous, the operator $H = -\Delta + V$ can be defined on twice differentiable functions with compact support (the set is dense in $L^2(\mathbb{R}^d)$). As it turns out, there exists a unique self-adjoint extension of this operator. This is the operator that we consider.

We now turn to the harmonic oscillator

$$H = \tfrac{1}{2}(-\Delta + x^2)$$

on $L^2(\mathbb{R})$. We know from Theorem 4.2 that the spectrum consists of isolated eigenvalues. We rewrite the Hamiltonian using creation and annihilation operators:

$$a^* = \tfrac{1}{\sqrt{2}}(X - iP)$$

$$a = \tfrac{1}{\sqrt{2}}(X + iP).$$

Exercise 8. Check that $[a, a^*] = 1$. Precisely, show that for any f, g in a dense set, we have

$$(f, aa^* g) - (f, a^* a g) = (f, g).$$

We have
$$H = a^* a + \tfrac{1}{2} = N + \tfrac{1}{2}$$

with $N = a^* a$.

Exercise 9. Check that
$$Na = a(N-1),$$
$$Na^* = a^*(N+1).$$