We start with $d=1$. The eigenvalue equation is

$$\frac{d^2}{dx^2} \psi(x) = -\lambda \psi(x), \quad \psi(x) = \psi(\xi) = 0.$$  

The solutions are $\psi_n(x) = \sin \left( \frac{2\pi n x}{b} \right)$, $n = \frac{n^2}{4} b$, $n \in \mathbb{N}$. They form an orthogonal basis of $L^2([0,b])$. Consequently, there are no other solutions, and the spectrum of $H$ is pure point, with $\sigma_p(H) = \left\{ \frac{n^2}{4} b : n \in \mathbb{N} \right\}$.

The solution of the problem for arbitrary $d$ follows. The solutions are

$$\psi_{n_1, \ldots, n_d}(x) = \frac{1}{(2\pi)^{d/2}} \sin \left( \frac{2\pi n_1 x_1}{b_1} \right),$$  

$$\sigma_{n_1, \ldots, n_d} = \frac{n^2}{4} b_1^2, \quad n_j \in \mathbb{N},$$  

for all $n_1, \ldots, n_d \in \mathbb{N}$. Since $\mathcal{D}(\mathcal{H}) = L^2([0,b^d])$, these functions form an orthogonal basis.

Another relevant potential is the finite square well, where

$$V(x) = \begin{cases} -V_0 & \text{if } x_j \in [0,b] \quad \forall j=1, \ldots, d, \\ 0 & \text{otherwise} \end{cases}$$

One can check that $\sigma_0(H) = [0, \infty)$ and $\sigma_0(H) = (-\infty, 0)$. See most textbooks, e.g. Gustafson & Sigal.

### 4.2 The harmonic oscillator

We again start with $d=1$. The Hilbert space is $L^2(\mathbb{R})$ and the Hamiltonian

$$H = -\Delta + \lambda.$$  

We consider a few general claims that allow to understand the properties of this class of Hamiltonians. The continuous spectrum of a self-adjoint operator can be characterized by the existence of a Weyl sequence.

**Definition 4.1.** Let $A$ be a self-adjoint operator on $\mathcal{H}$ and $\lambda$ be a complex number. A *Weyl sequence* for $(A, \lambda)$ is a sequence $(\psi_n)$ in $\mathcal{D}(A)$ such that

(i) $\| \psi_n \| = 1$ for all $n$.

(ii) $\| (A-\lambda) \psi_n \| \to 0$ as $n \to \infty$ (so $\psi_n$ is almost an eigenvector).

(iii) $\psi_n \to 0$ as $n \to \infty$. That is, $(\psi_n, \psi_n) \to 0$ for all $\psi \in \mathcal{H}$.

Then $(\psi_n)$ does not converge to an eigenvector.

**Theorem 4.1.** (Weyl). Let $A$ be self-adjoint on $\mathcal{H}$. Then $\lambda \in \sigma_0(A)$ iff there exists a Weyl sequence.

We skip the proof. We also accept the next result without proof.
Theorem 4.2. Let $V(x)$ be a continuous function on $\mathbb{R}^d$ such that $V(x) \to \infty$ as $|x| \to \infty$. Then the spectrum of $H = -\Delta + V$ consists of isolated eigenvalues $\lambda_n < \infty$ such that $2\lambda_n \to \infty$ as $n \to \infty$.

Although we do not prove it, we can reflect on why the theorem is true. We know that $\sigma(H) = \sigma_0(H) \cup \sigma_1(H)$. If $x \in \sigma_1(H)$, there exists a Weyl sequence $(\{n_k\})$, which converges weakly to 0. Then $\{n_k\}$ must converge to a Dirac, or have faster and faster oscillation, or disappear at infinity. (Or a combination of the three.) In the first two cases, $\|\Delta \{n_k\}\| \to \infty$; in the third case, $\|\{n_k\}\| \to \infty$. It is impossible for $\|\{-\Delta + V\} \{n_k\}\| \to \infty$ to stop bounded, let alone to vanish.

The proof that $2\lambda_n \to \infty$ follows from the minimax theorem.

We have eluded the mention of the domain of $H$. If $V$ is a "nice" function, such or continuous, the operator $H = -\Delta + V$ can be defined on twice differentiable functions with compact support (the set is dense in $L^2(\mathbb{R}^d)$). As it turns out, there exists a unique self-adjoint extension of this operator. This is the operator that we consider.

We now turn to the harmonic oscillator

$$H = \frac{1}{2} \left( -\Delta + x^2 \right)$$

on $L^2(\mathbb{R})$. We know from Theorem 4.2 that the spectrum consists of isolated eigenvalues. We rewrite the Hamiltonian using creation and annihilation operators:

$$a^* = \frac{1}{\sqrt{2}} \left( x - iP \right)$$

$$a = \frac{1}{\sqrt{2}} \left( x + iP \right).$$

Exercise 8. Check that $[a, a^*] = 1$. Precisely, show that for any $f, g$ in a dense set, we have

$$(f, a a^* g) = (f, a^* a g) = (f, g).$$

We have

$$H = a^* a + \frac{1}{2} = N + \frac{1}{2}$$

with $N = a a^*$. 

Exercise 9. Check that $Na = a(N - 1)$,

$$N a^* = a^* (N + 1).$$