Further exercises in integration

1. B. Use definite integrals to find the limits of the following sums:
   (a) \[ \lim_{n \to \infty} \left( \frac{1}{n^2} + \frac{2}{n^2} + \ldots + \frac{n-1}{n^2} \right) \]
   (b) \[ \lim_{n \to \infty} \left( \frac{1}{n+1} + \frac{1}{n+2} + \ldots + \frac{1}{n+n} \right) \]
   (c) \[ \lim_{n \to \infty} \frac{x^2 + 2^2 + \ldots + n^2}{np^{n+1}}, \quad p > 0 \]

2. A. Determine the sign of the following integral:
   \[ \int_0^{2\pi} f(x) \, dx, \]
   where \( f(0) = 1, \ f(x) = \frac{\sin(x)}{x} \) for \( x > 0 \).

3. A. Determine (without evaluating) which of the following integrals is greater:
   - \( \int_0^1 \sqrt{1 + x^2} \, dx \) or \( \int_0^1 x \, dx \)
   - \( \int_0^1 x^2 \sin^2(x) \, dx \) or \( \int_0^1 x \sin^2(x) \, dx \)
   - \( \int_1^2 e^x \, dx \) or \( \int_1^2 e^x \, dx \)

Uniform convergence.

4. B. Find \( f'(x) \) if
   \[ f(x) = \int_x^{10} e^{-xy^2} \, dy, \quad 0 \leq x \leq 10. \]
   Justify all steps of the calculation.

**Hint.** Let \( I(a, b, c) = \int_0^a dx f(x, c) \). You need to show that \( f \) satisfies all conditions necessary for FTC1 and the theorem on the differentiation of integrals depending on a parameter to hold. Then
   \[ I'(a(x), b(x), c(x)) = \frac{\partial I}{\partial a}(a(x), b(x), c(x))a'(x) \]
   \[ + \frac{\partial I}{\partial b}(a(x), b(x), c(x))b'(x) + \frac{\partial I}{\partial c}(a(x), b(x), c(x))c'(x), \]
   and each of the partial derivatives can be evaluated using either FTC1 or the theorem on the differentiation of integrals depending on a parameter.
5. B. Applying differentiation with respect to a parameter $\alpha$, evaluate the following integral

$$
\int_{0}^{\infty} \frac{e^{-\alpha x} - e^{-\beta x}}{x} \, dx, \; \alpha, \beta > 0.
$$

**Hint.** You may exchange the order of integration and differentiation without a justification. The exchange is justified due to the uniform convergence of the above improper integral depending on parameters. The theory of improper integrals depending on parameters is not currently covered by our course.

6. B. The Laplace transform of $f : [0, \infty) \to \mathbb{R}$ is a function $F : (0, \infty) \to \mathbb{R}$ defined by the formula

$$
F(p) = \int_{0}^{\infty} e^{-pt} f(t) \, dt, \; p > 0.
$$

Find the Laplace transform of the following functions:

(a) $f(t) = 1$

(b) $f(t) = e^{-\alpha t}, \; \alpha > 0$

(c) $f(t) = \cos(\beta t), \; \beta \in \mathbb{R}$

Notice that the Laplace transform of a bounded function is not necessarily bounded.

7. B. Prove the uniform convergence of the functional series in the indicated intervals:

(a) $\sum_{n=1}^{\infty} \frac{x^n}{n^2}, \; x \in [-1, 1]$

(b) $\sum_{n=1}^{\infty} \frac{\sin(nx)}{2n}, \; x \in \mathbb{R}$

8. B. Applying term-wise differentiation and integration, find the sums of the series in the indicated intervals:

(a) $\sum_{k=1}^{\infty} \frac{x^k}{k}, \; x \in [a, b], \; -1 < a < 0 < b < 1$

(b) $\sum_{k=1}^{\infty} (-1)^{k-1} \frac{x^k}{k}, \; x \in [a, b], \; -1 < a < 0 < b < 1$

(c) $\sum_{k=1}^{\infty} \frac{x^{2k-1}}{2k-1}, \; x \in [a, b], \; -1 < a < 0 < b < 1$.

Justify all steps of your calculations.

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