Assignment 7

Due Monday 28 November 15:00. Two problems will be selected for marking among those with boxed numbers. There are 4 points for each of the two problems, and 2 points for presentation. Do not forget to write your name and your group (1: Tuesday 12-1/Buze; 2: Tuesday 2-3/Vogel; 3: Thursday 1-2/Khor; 4: Friday 12-1/Matejczyk).

1. Consider the space of sequences \((x_k)_{k \geq 1}, x_k \in \mathbb{R}\), and the following norm:

\[
\|x\| = \sum_{k=1}^{\infty} 2^{-k} |x_k|.
\]

Show that \(\|\cdot\|\) is a norm.

2. Consider the space of sequences \((x_k)_{k \geq 1}, x_k \in \mathbb{R}\), and the following norm:

\[
\|x\| = \left( \sum_{k=1}^{\infty} k |x_k|^2 \right)^{1/2}.
\]

Show that \(\|\cdot\|\) is a norm. (Problems 5 and 6 may actually help.)

3. Are the norms in Problems 1 and 2 equivalent? Give justifications.

4. Let \(a = (a_k)_{k \geq 1}\) be a fixed sequence of real numbers, and let \(X = \{(x_k)_{k \geq 1} : \sum_k |a_k x_k| < \infty\}\). For \(x \in X\), define

\[
\|x\|_a = \sum_{k=1}^{\infty} a_k |x_k|.
\]

(a) Give sufficient and necessary conditions for \(a\) so that the function above is a norm.

Let \(a = (a_k)\) and \(b = (b_k)\) be two decreasing sequences of positive numbers.

(b) Assume that there exists \(\varepsilon > 0\) such that \(a_k/b_k \geq \varepsilon\) and \(b_k/a_k \geq \varepsilon\) for all \(k\). Prove that \(\|\cdot\|_a\) and \(\|\cdot\|_b\) are equivalent.

(c) Assume that \(a_k/b_k \to 0\) as \(k \to \infty\). Prove that \(\|\cdot\|_a\) and \(\|\cdot\|_b\) are not equivalent.

5. Check that the following are inner products:

(a) On \(\mathbb{R}^n\), let \(\langle x, y \rangle = \sum_{i=1}^{n} x_i y_i\).

(b) Let \(\ell\) be the space of sequences \((x_k)_{k \geq 1}\) such that \(\sum k |x_k|^2 < \infty\). Show that \(\langle x, y \rangle = \sum_{k=1}^{\infty} k x_k y_k\) is an inner product on \(\ell\).

6. Let \(\langle \cdot, \cdot \rangle\) be an inner product, and \(\|\cdot\| = \sqrt{\langle \cdot, \cdot \rangle}\) be its induced norm.

(a) Show that \(\|\cdot\|\) is a norm indeed.

(b) Show that \(\|\cdot\|\) satisfies the parallelogram identity.

(c) Show that the Euclidean norm on \(\mathbb{R}^n\), \(\|x\|_2 = \left(\sum_{i=1}^{n} |x_i|^2\right)^{1/2}\), satisfies the triangle inequality.