Assignment 2

Due Monday 23 October 15:00. Two problems will be selected for marking among those with boxed numbers. There are 4 points for each of the two problems, and 2 points for presentation. Do not forget to write your name and your group (1: Tuesday 12-1/Atkinson; 2: Tuesday 2-3/Vasdekis; 3: Thursday 1-2/Archer; 4: Friday 12-1/Bowditch).

1. (a) Let \( f(x) = x \sin \frac{1}{x} \) for \( x \in (0, 2\pi] \) and \( f(0) = 0 \). Prove that \( f \) is regulated.
(b) Let \( f(x) = \frac{1}{x} \) for \( x \in (0, 1] \) and \( f(0) = 0 \). Prove that \( f \) is not regulated.

2. (a) Give an example of a regulated function \( f \) on \([0, 1]\) such that \( f(x) \geq 0 \) for all \( x \in [0, 1] \), \( \int_0^1 f(x) \, dx = 0 \), and such that \( f(x_0) > 0 \) for at least one \( x_0 \in [0, 1] \).
(b) Show that if \( f \) is a continuous function on \([a,b]\) that satisfies \( f(x) \geq 0 \) for all \( x \in [a,b] \) and \( \int_a^b f(x) \, dx = 0 \), then \( f \) is identically zero.

3. Let \( f : [a,b] \to \mathbb{R} \) be regulated and non-negative. Prove that \( g : [a,b] \to \mathbb{R} \) defined by \( g(x) = \sqrt{f(x)} \) is regulated.

4. Let \( a > 0 \). By using a suitable sequence of step functions on \([0,a]\), show directly from the definition that \( \int_0^a \frac{1}{x} \, dx = \frac{1}{2}a^2 \).

5. By using a suitable sequence of step functions \( \varphi_n \) with partition \( P = \{0, \frac{1}{n}, \frac{2}{n}, ..., \frac{n-1}{n}, 1\} \), show directly from the definition that \( \int_0^1 e^x \, dx = e - 1 \).

6. The goal is to show the Riemann-Lebesgue lemma, namely that \( \int_a^b f(x) \sin(nx) \, dx \to 0 \) as \( n \to \infty \), for any regulated function \( f \). For this, show that
   (a) For all \( a < b \), we have \( \int_a^b \sin(nx) \, dx \to 0 \) as \( n \to \infty \).
   (b) By considering separately each interval of the partition, show that \( \int_a^b \varphi(x) \sin(nx) \, dx \to 0 \) as \( n \to \infty \) for all \( \varphi \in S[a,b] \).
   (c) Extend this to all \( f \in R[a,b] \).

7. Give an example of a function \( f : [0,1] \) that is bounded, piecewise continuous, and not regulated. Prove that it is impossible to approximate this function: There exists \( \varepsilon > 0 \) such that any step function \( \varphi \) satisfies \( \|f - \varphi\|_\infty > \varepsilon \).

8. Let \( f \in R[-1,1] \) be an odd function, meaning that \( f(-x) = -f(x) \). Show that \( \int_{-1}^1 f(x) \, dx = 0 \).