Assignment 4

Due Monday 6 November 15:00. Two problems will be selected for marking among those with boxed numbers. There are 4 points for each of the two problems, and 2 points for presentation. Do not forget to write your name and your group (1: Tuesday 12-1/Atkinson; 2: Tuesday 2-3/Vasdekis; 3: Thursday 1-2/Archer; 4: Friday 12-1/Bowditch).

1. This problem is motivated by the Gamma function \( \Gamma(x) = \int_0^\infty t^{x-1}e^{-t}dt \). For given \( x \in \mathbb{R} \), consider \( f(t) = t^{x-1}e^{-t} \).

   1. For which values of \( x \) does the integral \( \int_0^1 f(t)dt \) exist?
   2. For which values of \( x \) does the Riemann integral \( \int_0^1 f(t)dt \) exist?
   3. For which values of \( x \) does the improper integral \( \int_0^\infty f(t)dt \) exist?

2. Find the following integrals.

   1. \( \int_0^1 \frac{\cos \sqrt{t}}{\sqrt{t}} dt \).
   2. \( \int_2^\infty t^{-2} \log t dt \).
   3. \( \int_2^\infty e^{-\sqrt{t}} dt \).
   4. \( \int_0^\pi e^{\sin^2 t} \sin t \cos t dt \).

3. Let \( f(x) = \int_0^x e^{-t^2} dt \). Find its derivative \( f'(x) \), and draw it for \( x \in (-\infty, \infty) \). For approximately which \( x \) is \( f \) maximum?

4. Find the pointwise limits of the following functions as \( n \to \infty \). Is the convergence uniform? Prove it!

   1. \( f_n(x) = x^{1/n} \) for \( x \in [0, 1] \).
   2. \( f_n(x) = \sin(x + \frac{1}{n}) \) for \( x \in \mathbb{R} \).
   3. \( f_n(x) = e^{n(\cos x - 1)} \) for \( x \in \mathbb{R} \).
   4. \( f_n(x) = e^{x/n} \) for \( x \in [0, 2\pi] \).

5. Find the pointwise limits of the following functions as \( n \to \infty \). Is the convergence uniform? Prove it!

   1. \( f_n(x) = \min(\cos x, 1 - \frac{1}{n}) \) for \( x \in \mathbb{R} \).
   2. \( f_n(x) = n \sin \frac{x}{n} \) for \( x \in \mathbb{R} \).
   3. \( f_n(x) = e^{-x/n} \) for \( x \in [0, \infty) \).
   4. \( f_n(x) = \lim_{m \to \infty} [\cos(n! \pi x)]^{2m} \) for \( x \in [0, 1] \).

6. Consider the functions \( f_n(x) = n^a x e^{-n^b x} \) on \( [0, \infty) \), where \( a, b \) are fixed numbers. Find the pointwise limit as \( n \to \infty \). Draw a few functions, calculate the derivatives, and find the values of \( a, b \) for which the convergence is uniform.