Assignment 5

Due Monday 13 November 15:00. Two problems will be selected for marking among those with boxed numbers. There are **4 points** for each of the two problems, and **2 points** for presentation. Do not forget to write **your name** and **your group** (**1**: Tuesday 12-1/Atkinson; **2**: Tuesday 2-3/Vasdekis; **3**: Thursday 1-2/Archer; **4**: Friday 12-1/Bowditch).

1. Let f be the following function on $[0,2] \times [0,1]$:

$$f(x,y) = \begin{cases} \frac{xy(x^2-y^2)}{(x^2+y^2)^3} & \text{if } (x,y) \neq (0,0), \\ 0 & \text{if } x = y = 0. \end{cases}$$

Compute $\displaystyle\int_0^2 \mathrm{d}x \int_0^1 \mathrm{d}y\, f(x,y)$ and $\displaystyle\int_0^1 \mathrm{d}y \int_0^2 \mathrm{d}x\, f(x,y)$.

2. Prove that the Euler integrals of the first kind (beta-function) converge when $p > 0$ and $q > 0$:

$$B(p,q) = \int_0^1 x^{p-1}(1-x)^{q-1}\mathrm{d}x.$$

3. Let $I_n = \displaystyle\int_0^{\pi/2} \cos^n(x)\mathrm{d}x$, for $n \in \mathbb{N}$. Use integration by parts to show that

$$I_n = \frac{n-1}{n} I_{n-2},$$

for all $n \geq 2$. Use this to find an explicit expression for I_n, for arbitrary n. (Notation: For $n \in \mathbb{N}$, $n!! = n(n-2)(n-4)...$, ending at 1 or 2, depending on whether n is odd or even.)

4.

(a) Show that $\sin t \leq t$ for all $t \geq 0$. For this, define $g(t) = t - \sin t$; use the fundamental theorem of calculus to write $g(t) = \int_0^t g'(s)\mathrm{d}s$, and check that $g'(s) \geq 0$ for all s.

(b) Show that $\displaystyle\int_0^\infty x^{-a} \sin \tfrac{1}{x}\, \mathrm{d}x$ exists as an improper integral for $a \in (0,1)$.

(c) Show that $\displaystyle\int_0^\infty x^{-a} \cos \tfrac{1}{x}\, \mathrm{d}x$ exists as an improper integral for $a \in (1,2)$.

(Hint: Use integration by parts to get an expression involving $\int_0^\infty x^{-b} \sin \frac{1}{x}\, \mathrm{d}x$ for some b, and use (b).)

5. Use integration by parts to show that $\int_0^\pi f(x)\sin(nx)\,\mathrm{d}x$ goes to 0 as $n \to \infty$, for any differentiable function f.

6. The devil's staircase as limit of differentiable functions. Let $f_0(x) = x^2[1-(1-x)^2]$, and for $n \geq 0$ and $x \in [0,1]$, let

$$f_{n+1}(x) = \begin{cases} \frac{1}{2} f_n(3x) & \text{if } x \in [0, \frac{1}{3}), \\ \frac{1}{2} & \text{if } x \in [\frac{1}{3}, \frac{2}{3}], \\ \frac{1}{2} + \frac{1}{2} f_n(3x-2). \end{cases}$$

(a) Check that f_0 is differentiable, increasing, and that $f(0) = 0, f(1) = 1, f'(0) = f'(1) = 0$.

(b) Draw f_0, f_1, and f_2 on the interval $[0,1]$.

(c) Show by induction that f_n is continuous and differentiable on $[0,1]$.

(d) Check that $(f_n)_{n\geq1}$ is a Cauchy sequence with respect to the sup norm.

Can you conclude that the limiting function $f = \lim_n f_n$ exists and is differentiable?