Assignment 5

Due Monday 13 November 15:00. Two problems will be selected for marking among those with boxed numbers. There are 4 points for each of the two problems, and 2 points for presentation. Do not forget to write your name and your group (1: Tuesday 12-1/Atkinson; 2: Tuesday 2-3/Vasdekis; 3: Thursday 1-2/Archer; 4: Friday 12-1/Bowditch).

1. Let $f$ be the following function on $[0, 2] \times [0, 1]$:
   \[ f(x, y) = \begin{cases} \frac{xy(x^2-y^2)}{(x^2+y^2)^3} & \text{if } (x, y) \neq (0, 0), \\ 0 & \text{if } x = y = 0. \end{cases} \]
   Compute $\int_0^2 dx \int_0^1 dy f(x, y)$ and $\int_0^1 dy \int_0^2 dx f(x, y)$.

2. Prove that the Euler integrals of the first kind (beta-function) converge when $p > 0$ and $q > 0$:
   \[ B(p, q) = \int_0^1 x^{p-1}(1-x)^{q-1}dx. \]

3. Let $I_n = \int_0^{\pi/2} \cos^n(x)dx$, for $n \in \mathbb{N}$. Use integration by parts to show that
   \[ I_n = \frac{n-1}{n} I_{n-2}, \]
   for all $n \geq 2$. Use this to find an explicit expression for $I_n$, for arbitrary $n$. (Notation: For $n \in \mathbb{N}$, $n!! = (n-2)(n-4)\ldots$, ending at 1 or 2, depending on whether $n$ is odd or even.)

4. (a) Show that $\sin t \leq t$ for all $t \geq 0$. For this, define $g(t) = t - \sin t$; use the fundamental theorem of calculus to write $g(t) = \int_0^t g'(s)ds$, and check that $g'(s) \geq 0$ for all $s$.
   (b) Show that $\int_0^\infty x^{-a} \sin x dx$ exists as an improper integral for $a \in (0, 1)$.
   (c) Show that $\int_0^\infty x^{-a} \cos x dx$ exists as an improper integral for $a \in (1, 2)$.
      (Hint: Use integration by parts to get an expression involving $\int_0^\infty x^{-b} \sin \frac{1}{x} dx$ for some $b$, and use (b).)

5. Use integration by parts to show that $\int_0^\pi f(x) \sin(nx) dx$ goes to 0 as $n \to \infty$, for any differentiable function $f$.

6. The devil’s staircase as limit of differentiable functions. Let $f_0(x) = x^2[1-(1-x)^2]$, and for $n \geq 0$ and $x \in [0, 1]$, let
   \[ f_{n+1}(x) = \begin{cases} \frac{1}{2} f_n(3x) & \text{if } x \in [0, \frac{1}{3}), \\ \frac{1}{2} (1 + \frac{1}{2} f_n(3x - 2)) & \text{if } x \in [\frac{1}{3}, \frac{2}{3}], \\ \frac{1}{2} + \frac{1}{2} f_n(3x - 2) & \text{if } x \in [\frac{2}{3}, 1]. \end{cases} \]
   (a) Check that $f_0$ is differentiable, increasing, and that $f(0) = 0, f(1) = 1, f'(0) = f'(1) = 0$.
   (b) Draw $f_0, f_1$, and $f_2$ on the interval $[0, 1]$.
   (c) Show by induction that $f_n$ is continuous and differentiable on $[0, 1]$.
   (d) Check that $(f_n)_{n \geq 1}$ is a Cauchy sequence with respect to the sup norm.
   Can you conclude that the limiting function $f = \lim_{n \to \infty} f_n$ exists and is differentiable?