Infinite-volume setting: observables, interactions, states

Context: For $\Lambda \Subset \mathbb{Z}^d$, we have

- Hilbert space $\mathcal{H}_\Lambda = \bigotimes_{x \in \Lambda} \mathbb{C}^{2J+1} \simeq \ell^2(\Omega_\Lambda)$

- Hamiltonian: $H_\Lambda = \sum_{X \subset \Lambda} \Phi_X$

- Gibbs state: $\langle A \rangle = \frac{1}{Z} \operatorname{Tr} A e^{-\beta H_\Lambda}$

As $\Lambda \nearrow \mathbb{Z}^d$: $\mathcal{H}_{\mathbb{Z}^d}$ not useful (not separable!)

$H_{\mathbb{Z}^d}$ not defined.

Classical spin systems: $\Omega_{\mathbb{Z}^d}$, DLR measures.

Does not work here.

Local observables

Given $X \Subset \mathbb{Z}^d$, let $\mathcal{H}_X = \bigotimes_{x \in \Lambda} \mathbb{C}^{2J+1}$

$$\mathfrak{A}_X = \mathcal{B}(\mathcal{H}_X)$$

If $X \subset Y$: injection $\iota : \mathfrak{A}_X \longrightarrow \mathfrak{A}_Y$

$$\iota A = A \otimes \mathbb{1}_{Y \setminus X}$$

Exercise: Show that $\| \iota A \| = \| A \|$.

Algebra of local observables: inductive union $\mathfrak{A}_{loc} = \bigvee_{X \Subset \mathbb{Z}^d} \mathfrak{A}_X$.

Algebra of quasi-local observables: $\mathfrak{A} = \overline{\mathfrak{A}_{loc}}^{\,norm}$, the norm completion.

Then if $A \in \mathfrak{A}$, $\exists$ Cauchy sequence $A_n \in \mathfrak{A}_{X_n}$ s.t. $\| A - A_n \| \to 0$ as $n \to \infty$.

Remark: $\mathcal{A}$ has the structure of a Banach C^* algebra.

Translations: $\tau_y : \mathcal{H}_X \longrightarrow \mathcal{H}_{X+y}$

$$\tau_y \bigotimes_{x \in X} \varphi_x = \bigotimes \varphi_{x+y}$$

This extends to operators:

$$\tilde{\tau}_y : \mathcal{B}(\mathcal{H}_X) \longrightarrow \mathcal{B}(\mathcal{H}_{X+y})$$

$\tilde{\tau}_y A$ is the operator such that

$$(\tilde{\tau}_y A)\, \varphi = \tau_y \left(A\, \tau_y^{-1} \varphi \right)$$

The translation is a **-automorphism*:

$$\tau_y (AB) = \tau_y A \; \tau_y B \quad \text{and} \quad (\tau_y A)^* = \tau_y A^*.$$

Interactions

An interaction Φ is collection $(\Phi_X)_{X \in \mathbb{Z}^d}$ where $\Phi_X \in \mathcal{A}_X$.

An interaction is translation-invariant if $\tau_y \Phi_X = \Phi_{X+y}$

All our interactions are translation-invariant.

Two norms of interactions:

$$|||\Phi||| = \sum_{X \ni 0} \frac{\|\Phi_X\|}{|X|} \qquad (\text{then } \Phi \in \mathcal{I})$$

$$\|\Phi\|_r = \sum_{X \ni 0} \|\Phi_X\| e^{r|X|}, \quad r \geq 0 \qquad (\text{then } \Phi \in \mathcal{I}_r)$$

Hamiltonians & finite-volume Gibbs states

Given $\Phi \in \mathcal{I}$ and $\Lambda \Subset \mathbb{Z}^d$, let

$$H_\Lambda^{\Phi} = \sum_{X \subset \Lambda} \Phi_X$$

$$\langle A \rangle_{\Lambda,\beta}^{\Phi} = \frac{1}{Z_{\Lambda,\beta}(\Phi)} \operatorname{Tr}_{\mathcal{H}_\Lambda} A \, e^{-\beta H_\Lambda^{\Phi}}$$

Infinite-volume states

These are states on $\mathfrak{A}$ (the algebra of quasilocal obs.)

DEFINITION 3.4. *A* **state** $\langle\cdot\rangle$ *is a normalised, positive linear functional on* $\mathcal{A}$. *That is,* $\langle\cdot\rangle$ *satisfies*

(i) $\langle sA + tB\rangle = s\langle A\rangle + t\langle B\rangle$ *for all* $A, B \in \mathcal{A}$ *and* $s, t \in \mathbb{C}$.

(ii) $\langle \mathbb{1}\rangle = 1$.

(iii) $\langle A^*A\rangle \geq 0$ *for all* $A \in \mathcal{A}$.

We write $\mathfrak{E}$ *for the set of states. A state is called* **translation-invariant** *if* $\langle A\rangle = \langle \tau_x A\rangle$ *for all* $x \in \mathbb{Z}^d$.

Exercise: States have norm 1.

Gibbs state:

DEFINITION 3.5 (State as cluster point). *Let* $\Phi \in \mathcal{I}$, *and let* Ψ_n *be a sequence of interactions in* $\mathcal{I}$ *such that* $\|\Psi_n\| \to 0$ *as* $n \to \infty$. *Let* $\Lambda_n = \{-n, \ldots, n\}^d$ *and for* $A \in \mathcal{A}_{\text{loc}}$, *let*

$$\langle A\rangle^{\Phi}_{\Lambda_n,\beta} = \frac{1}{Z_{\Lambda_n,\beta}(\Phi + \Psi_n)} \operatorname{Tr} A\, \mathrm{e}^{-\beta H^{\Phi+\Psi_n}_{\Lambda_n}}.$$

The cluster points of the sequence $(\langle\cdot\rangle^{\Phi}_{\Lambda_n,\beta})_{n\geq 1}$ *are infinite-volume Gibbs states for the interaction* Φ.

Examples

Ising model:

$$\Phi_X = \begin{cases} -S_x^{(3)} S_y^{(3)} & \text{if } X = \{x,y\} \in \mathcal{E}_{\mathbb{Z}^d} \\ -h\, S_x^{(3)} & \text{if } X = \{x\} \\ 0 & \text{oth.} \end{cases}$$

The H_Λ^Φ is the Ising hamiltonian with free b.c.

Heisenberg model:

$$\Phi_X = \begin{cases} -\vec{S}_x \cdot \vec{S}_y & \text{if } X = \{x,y\} \in \mathcal{E}_{\mathbb{Z}^d} \\ 0 & \text{oth.} \end{cases}$$

Main goals & relevant questions

A model is given by an interaction $\Phi = (\Phi_X)$ or a family of interactions (Φ_X^h).

- For given $\beta > 0$, how many infinite-volume Gibbs states?
- Properties of the Gibbs states, and of the correlation functions?

Partial answers:

- A common general theory (extremal states, ...).
- More detailed results for specific models.

First step: Various characterisations of ∞-vol. Gibbs states.

Recall: finite fixed domain:

$$\langle A \rangle_{H,\beta} = \frac{1}{Z_\beta} \operatorname{Tr} A \, e^{-\beta H}.$$

PROPOSITION 3.3. *The Gibbs state $\langle\cdot\rangle_{H,\beta}$ is the unique state $\langle\cdot\rangle = \operatorname{Tr}\cdot\rho$ satisfying any of the following four conditions:*

(a) Tangent condition: *Let $F_\beta(H) := -\frac{1}{\beta}\log\operatorname{Tr}\mathrm{e}^{-\beta H}$. Then*

$$F_\beta(H+A) \le F_\beta(H) + \langle A\rangle$$

for all $A = A^* \in \mathcal{B}(\mathcal{H})$.

(b) Gibbs variational principle: *The density matrix ρ minimizes the function*

$$\mathcal{F}_\beta(\rho) := \operatorname{Tr} H\rho + \tfrac{1}{\beta}\operatorname{Tr}\rho\log\rho.$$

(c) KMS condition: *Define the time evolution*

$$\alpha_t(A) = \mathrm{e}^{\mathrm{i}tH} A\,\mathrm{e}^{-\mathrm{i}tH}, \qquad t \in \mathbb{C}. \tag{3.11}$$

Then

$$\langle AB\rangle = \langle B\,\alpha_{\mathrm{i}\beta}(A)\rangle$$

for all $A, B \in \mathcal{B}(\mathcal{H})$.

(d) RAS condition: *For all $A \in \mathcal{B}(\mathcal{H})$ such that $\langle AA^*\rangle > 0$,*

$$\langle A^*[H,A]\rangle \ge \frac{1}{\beta}\langle A^*A\rangle \log\frac{\langle A^*A\rangle}{\langle AA^*\rangle} \tag{3.12}$$

All characterisations can be generalised to infinite volumes!

Here: (a) and (c).

Finite-volume free energy

Free energy as function of the interactions:

$$f_\Lambda(\Phi, \beta) = -\frac{1}{\beta|\Lambda|} \log \operatorname{tr} \mathrm{e}^{-\beta H_\Lambda^\Phi} \, .$$

PROPOSITION 3.7.

(a) *The free energy f_Λ is a concave function of the interactions.*

(b) *$(f_\Lambda)_{\Lambda \Subset \mathbb{Z}^d}$ are equicontinuous: for any $\Phi, \Phi' \in \mathcal{I}$ we have*

$$\big| f_\Lambda(\Phi, \beta) - f_\Lambda(\Phi', \beta) \big| \leq \|\Phi - \Phi'\|.$$

Proof:

Infinite-volume limit of $f_\Lambda(\Phi, \beta)$

Definition 3.8. *A sequence of finite domains $(\Lambda_n)_{n\geq 1}$ converges to $\mathbb{Z}^d$* **in the sense of van Hove** *if*

(i) it is increasing: $\Lambda_{n+1} \supset \Lambda_n$ for all n;
(ii) it invades $\mathbb{Z}^d$: $\cup_{n\geq 1}\Lambda_n = \mathbb{Z}^d$;
(iii) the ratio boundary/bulk vanishes: $\frac{|\partial_r \Lambda_n|}{|\Lambda_n|} \to 0$ as $n \to \infty$, $\forall\, r$.

Here, the r-boundary is $\partial_r \Lambda = \{x \in \Lambda^c : \mathrm{dist}(x, \Lambda) \leq r\}$.

Theorem 3.9. *Assume that $\Phi \in \mathcal{I}$, i.e. $\|\Phi\| < \infty$. Then the limit*

$$f(\Phi, \beta) := \lim_{n\to\infty} f_{\Lambda_n}(\Phi, \beta)$$

exists and is the same along all van Hove sequences $\Lambda_n \Uparrow \mathbb{Z}^d$. It is a concave function of the interactions.

Fundamental result in Statistical Mechanics!

This is a gate from the microscopic to the macroscopic World!

For the proof, we use:

LEMMA A.12. *Assume that $(a_\Lambda)_{\Lambda \Subset \mathbb{Z}^d}$ is a set-indexed sequence of real numbers that satisfies*

- *translation invariance: $a_{\Lambda+x} = a_\Lambda$ for all $\Lambda \Subset \mathbb{Z}^d$ and $x \in \mathbb{Z}^d$;*
- *bulk property: there exists a constant c such that for any k and any mutually disjoint $\Lambda_1, \ldots, \Lambda_k \Subset \mathbb{Z}^d$, we have*

$$\left| a_{\cup_{i=1}^k \Lambda_i} - \sum_{i=1}^k a_{\Lambda_i} \right| \leq c \sum_{i=1}^k |\partial_1 \Lambda_i|.$$

Then there exists $\lambda \in \mathbb{R}$ such that

$$\lambda = \lim_{n\to\infty} \frac{a_{\Lambda_n}}{|\Lambda_n|}$$

along any van Hove sequence (Λ_n) of domains in $\mathbb{Z}^d$.

Proof of Theorem 3.9:

Gibbs states as tangent functional

Given $\Psi \in \mathcal{I}$, let $A_\Psi = \sum_{X \ni 0} \frac{1}{|X|} \Psi_X$.

Note: $\|A_\Psi\| \leq |||\Psi|||$.

Definition 3.10 (States as tangent functionals). *A translation-invariant state $\langle\cdot\rangle$ on $\mathcal{A}$ is an equilibrium state for the interaction Φ, in the sense of tangent functionals to the free energy, if*

$$f(\Phi + \Psi) \leq f(\Phi) + \langle A_\Psi \rangle$$

for all $\Psi \in \mathcal{I}$.

Remark: We saw that $\langle \cdot \rangle^{\Phi}_{\Lambda,\beta} = \frac{1}{Z^{\Phi}_{\Lambda,\beta}} \operatorname{Tr} \cdot e^{-\beta H^{\Phi}_{\Lambda}}$ satisfies

$$\boxed{f_\Lambda(\Phi + \Psi, \beta) \leq f_\Lambda(\Phi, \beta) + \underbrace{\langle \tfrac{1}{|\Lambda|} H^{\Psi}_{\Lambda} \rangle^{\Phi}_{\Lambda,\beta}}_{}.}$$

$\longrightarrow \langle A_\Psi \rangle$ as $\Lambda \Uparrow \mathbb{Z}^d$.

The "cluster point Gibbs states" are then tangent states!

KMS states

Recall:

Proposition 3.3. *The Gibbs state $\langle\cdot\rangle_{H,\beta}$ is the unique state $\langle\cdot\rangle = \mathrm{Tr}\cdot\rho$ satisfying any of the following four conditions:*

(a) Tangent condition: *Let $F_\beta(H) := -\frac{1}{\beta}\log \mathrm{Tr}\, \mathrm{e}^{-\beta H}$. Then*

$$F_\beta(H+A) \leq F_\beta(H) + \langle A\rangle$$

for all $A = A^* \in \mathcal{B}(\mathcal{H})$.

(b) Gibbs variational principle: *The density matrix ρ minimizes the function*

$$\mathcal{F}_\beta(\rho) := \mathrm{Tr}\, H\rho + \tfrac{1}{\beta}\mathrm{Tr}\,\rho\log\rho.$$

(c) KMS condition: *Define the time evolution*

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Then

$$\langle AB\rangle = \big\langle B\,\alpha_{\mathrm{i}\beta}(A)\big\rangle$$

for all $A, B \in \mathcal{B}(\mathcal{H})$.

(d) RAS condition: *For all $A\in\mathcal{B}(\mathcal{H})$ such that $\langle AA^*\rangle > 0$,*

$$\langle A^*[H,A]\rangle \geq \frac{1}{\beta}\langle A^*A\rangle \log\frac{\langle A^*A\rangle}{\langle AA^*\rangle} \tag{3.12}$$

Given an interaction $\Phi \in \mathcal{I}$, let

$$\alpha^{\Phi}_{\Lambda,t}(A) = \mathrm{e}^{\mathrm{i}tH^{\Phi}_{\Lambda}}\, A\, \mathrm{e}^{-\mathrm{i}tH^{\Phi}_{\Lambda}}.$$

(for $A \in \mathcal{A}_\Lambda$.)

We first show the existence of the limit $\Lambda \Uparrow \mathbb{Z}^d$.

Infinite-volume limit of the evolution operator

PROPOSITION 3.15 (Infinite-volume limit of the evolution operator). *Assume that $\Phi \in \mathcal{I}_r$ for some $r > 0$ and that $t \in \mathbb{R}$. There exists a $*$-automorphism $\alpha_t^\Phi : \mathcal{A} \to \mathcal{A}$ such that*

$$\lim_{\Lambda \Uparrow \mathbb{Z}^d} \|\alpha_{\Lambda,t}^\Phi(A) - \alpha_t^\Phi(A)\| = 0 \quad \textit{for all } A \in \mathcal{A}_{\mathrm{loc}}. \tag{3.39}$$

Further $\|\alpha_t^\Phi\| = 1$ and α_t^Φ satisfies the group property

$$\alpha_{s+t}^\Phi(A) = \alpha_s^\Phi\big(\alpha_t^\Phi(A)\big) \qquad \textit{for all } A \in \mathcal{A}, s, t \in \mathbb{R}.$$

Proof:

(1) For $|t| < \frac{r}{2\|\Phi\|_r}$, we show that $(\alpha_{\Lambda,t}^\Phi)_{\Lambda \Subset \mathbb{Z}^d}$ is Cauchy for each fixed $A \in \mathcal{A}_{\mathrm{loc}}$. We denote the limit $\alpha_t^\Phi(A)$.

(2) For $t \in \mathbb{R}$, we have $\|\alpha_{\Lambda,t}^\Phi(A)\| = \|A\|$ for all Λ, so $\|\alpha_t^\Phi\| = 1$.

(3) We use the group property to extend α_t^Φ to the whole real line.