Uniqueness / non-uniqueness of Gibbs states

Clarification for last week: decomposition of states

- We need $\mathcal{G}^{\Phi}$ to be a metrisable set in order to apply Choquet's theorem.
- The normalised linear functionals on $\mathfrak{A}$ are compact in weak $*$ topology (Banach-Alaoglu).
- The subset of Gibbs states is closed, so it is compact.
- The weak $*$ topology is not metrisable in general, but compact subsets are metrisable.
- Since we deal with weak $*$ topology, the claim $\langle \cdot \rangle = \int \gamma(\cdot)\, \mu(d\gamma)$ means that $\forall A \in \mathfrak{A}$, we have $\langle A \rangle = \int \gamma(A)\, \mu(d\gamma)$.

Recall basic models & their phase diagrams

$$H_{\Lambda,h} = -\sum_{xy\in\mathcal{E}_\Lambda} \left(J^1 S^1_x S^1_y + J^2 S^2_x S^2_y + J^3 S_x S_y \right) - h \sum_{x\in\Lambda} S^3_x$$

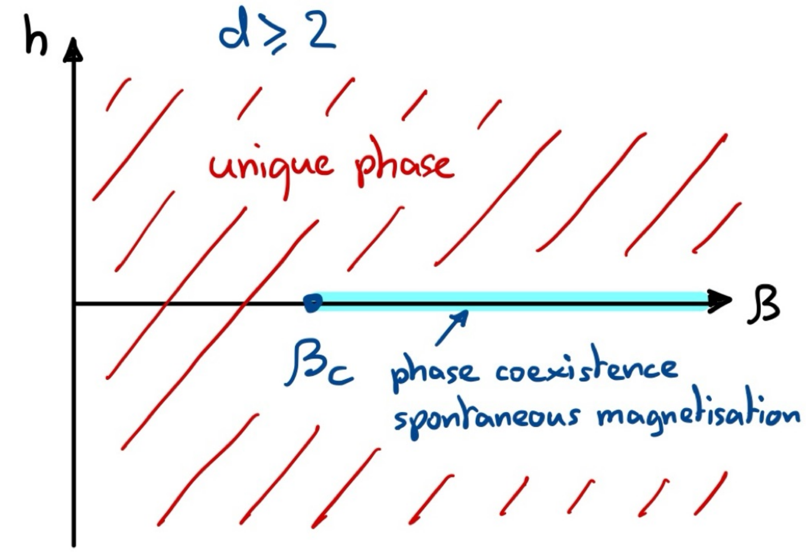


FIGURE 1.1. Phase diagram of the **Ising model**, and of the **XXZ model** for $J^{(3)} > J^{(1)} = J^{(2)} \geq 0$, for all dimensions $d \geq 2$.

$$H_{\Lambda,h} = -\sum_{xy\in\mathcal{E}_\Lambda} \left(J^1 S^1_x S^1_y + J^2 S^2_x S^2_y + J^3 S_x S_y \right) - h \sum_{x\in\Lambda} S^3_x$$

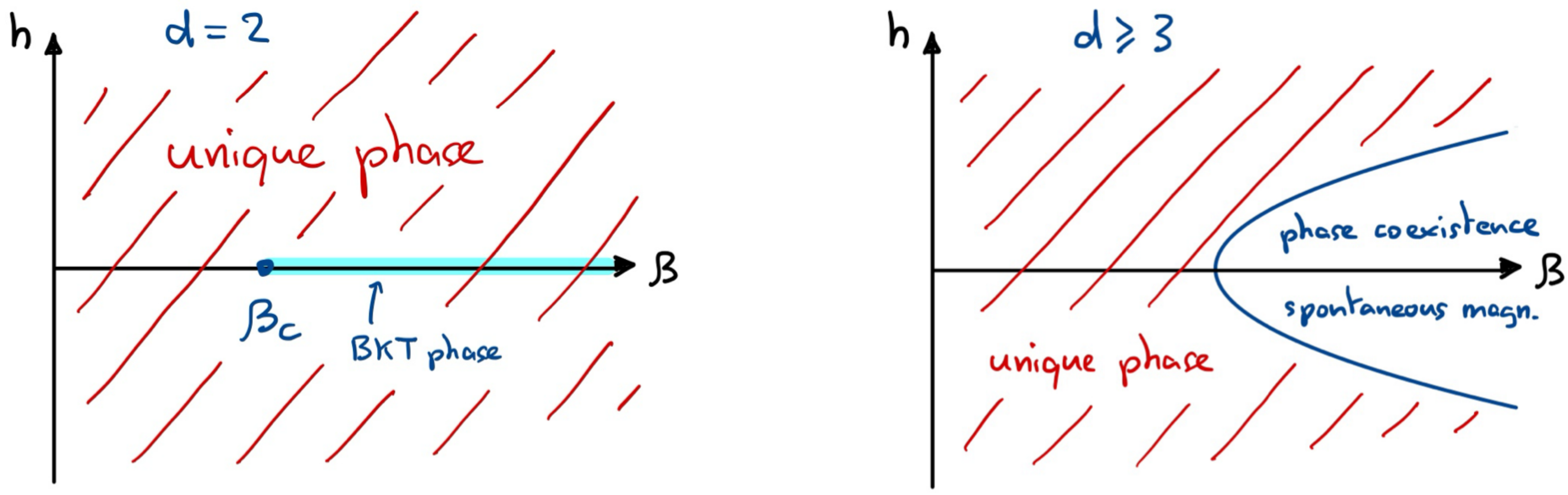


FIGURE 1.2. Phase diagrams of the **XXZ model** for $J^{(1)} = J^{(2)} > |J^{(3)}|$. The BKT phase is the Berezinsky-Kosterlitz-Thouless phase where the two point correlation function has power-law decay (in the unique phase, decay is exponential).

$$H_{\Lambda,h} = - \sum_{xy \in \mathcal{E}_\Lambda} \left(J^1 S^1_x S^1_y + J^2 S^2_x S^2_y + J^3 S_x S_y \right) - h \sum_{x \in \Lambda} S^3_x$$

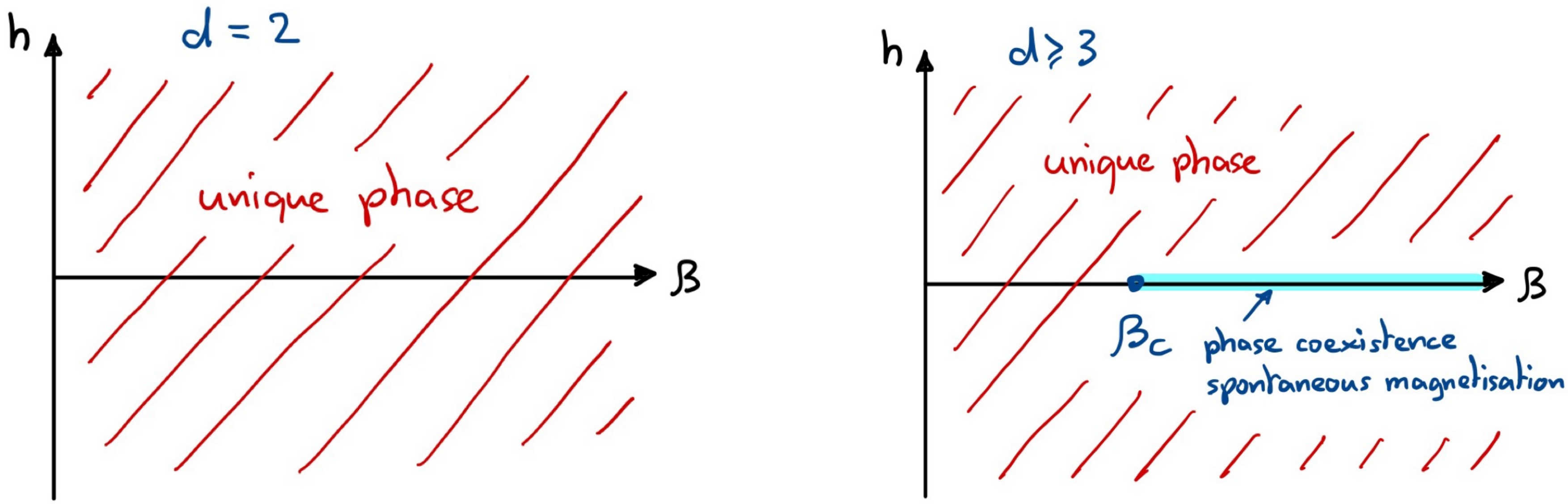


FIGURE 1.3. Phase diagrams of the **Heisenberg** or **XXX model** for which $J^{(1)} = J^{(2)} = J^{(3)} \geq 0$.

Next: a sufficient condition for uniqueness.

A sufficient condition for uniqueness – high temperature

THEOREM 4.1. *Assume that there exists $r > 0$ such that $\|\Phi\|_r < \frac{r}{2}$ and that*

$$\frac{\|\Phi\|_r}{r - 2\|\Phi\|_r} < \mathrm{e}^{-r} \,.$$

Then the KMS state for the interaction Φ is unique.

Remark: If we include β, the condition is that $\beta \|\Phi\|_r$ is small.

Method of proof: This form of the KMS condition:

$$\rho(\underbrace{[A, B]}_{AB - BA}) = \rho(\underbrace{B(\alpha_i^{\Phi} - \mathbb{1})A}_{B\alpha_i(A) - BA})$$

$$\|\Phi\|_r = \sum_{X \ni 0} e^{r(|X|-1)} \|\Phi_X\|$$

$$\|\Phi\|_r = \sup_{y \in \mathbb{Z}^d} \sum_{X \ni y} e^{r(|X|-1)} \|\Phi_X\|$$

$$\langle AB \rangle = \langle B\alpha_i(A) \rangle$$

$$\mathrm{tr}\,\cdot = \frac{1}{N^{|\Lambda|}} \mathrm{Tr}\,\cdot$$

Also useful: Hilbert-Schmidt inner product:

$A, B \in \mathfrak{A}_\Lambda$: $\langle A, B \rangle_{HS} = \mathrm{tr}\, A^* B$

The induced norm is $\|A\|_2 = \sqrt{\mathrm{tr}\, A^* A}$.

Basis for the space of ~~hermitian~~ matrices on $\mathbb{C}^N$:

$$\left(e^{(m)}\right)_{m=0,\dots,N^2-1}$$

$\begin{pmatrix} 0 & & & \\ & 1 & & 0 \\ & & -1 & \\ 0 & & & 0 \end{pmatrix} \leftarrow i$

$$e^{(0)} = \mathbb{1};$$

$$|i\rangle\langle i| - |i+1\rangle\langle i+1| \quad \text{for } i = 1, \dots, N-1;$$

$$|i\rangle\langle j| + |j\rangle\langle i| \quad \text{for } 1 \le i < j \le N;$$

$$-\mathrm{i}|i\rangle\langle j| + \mathrm{i}|j\rangle\langle i| \quad \text{for } 1 \le i < j \le N.$$

$\mathrm{Tr}\begin{pmatrix} 0 & & \\ & 1 & \\ & & -1 \end{pmatrix}\begin{pmatrix} 1 & \\ & -1 \end{pmatrix}$

j

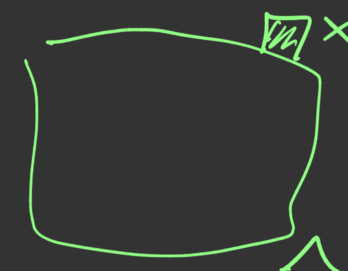
i

$i \to \begin{pmatrix} & 1 \\ 1 & \end{pmatrix}$

Then $$e^{(m)} = \frac{i}{2}\left[a^{(m)}, b^{(m)}\right] \qquad m \geq 1$$

where $$\|e^{(m)}\| = \|a^{(m)}\| = \|b^{(m)}\| = 1$$

$$\|e^{(m)}\|_2 = \|a^{(m)}\|_2 = \|b^{(m)}\|_2 = \sqrt{\frac{2}{N}}$$



$\mathcal{A}^{(N^2-1)}_{\Lambda,x} = \mathcal{A}_{\Lambda\cup\{x\}}$

<u>Def.</u>: $\mathcal{A}^{(m)}_{\Lambda,x}$: operators of the form $A \otimes B$

$A \in \mathcal{A}_\Lambda$, B "$= \sum_{j=0}^{m} c_j e_x^{(j)}$"

(~~real~~ linear space)

Proof of uniqueness theorem.

Let $\rho^{(0)}, \rho$ be two KMS states. We show that

$$\rho^{(0)}(A) = \rho(A) \quad \forall A \in \mathfrak{A}^{(m)}_{\Lambda,x} \quad , \quad \forall \Lambda, x, m$$

By induction on Λ, m.

Case $\Lambda = \emptyset, \; x = 0, \; m = 0$: $\quad \mathfrak{A}^{(0)}_{\emptyset,0} = \{ c\mathbb{1} : c \in \overset{\mathbb{C}}{\cancel{\mathbb{R}}} \}$

Then $\rho(c\mathbb{1}) = c = \rho^{(0)}(c\mathbb{1})$.

We now consider $\mathfrak{A}^{(m+1)}_{\Lambda,x}$.

$$A=\sum_{j=1}^{k}\underbrace{A_j}_{\in\mathcal{A}_\Lambda^{\cancel{st}}}\otimes\underbrace{B_j}_{\in\mathcal{A}_{\emptyset,x}^{(m+1)}}=\underbrace{\sum_{j=1}^{k}A_j\otimes B_j'}_{\equiv A'\in\mathcal{A}_{\Lambda,x}^{(m)}}+\underbrace{\sum_{j=1}^{k}c_jA_j}_{\equiv A''\in\mathcal{A}_\Lambda}\otimes e_x^{(m+1)}$$

$$=A'+A''\otimes e_x^{(m+1)}.$$

$A\in\Omega_{\Lambda,x}^{(m+1)}$

$$\rho(A)=\rho(A')+\rho(A''\otimes e_x^{(m+1)})$$

$\frac{1}{2}[\mathbb{1}\otimes a_x^{(m+1)},A''\otimes b_x^{(m+1)}]$

$$=\rho^{(0)}(A')+\rho(\tfrac{1}{2}A''\otimes[a_x^{(m+1)},b_x^{(m+1)}])$$

$$\overset{\text{KMS}}{=}\rho^{(0)}(A')+\rho\big(\tfrac{1}{2}(A''\otimes b_x^{(m+1)})(\alpha_{\mathrm{i}}-\mathbb{1})(\mathbb{1}_\Lambda\otimes a_x^{(m+1)})\big).$$

KMS: $\rho([A,B])=\rho(B(\alpha_i-\mathbb{1})A)$

Let $\rho_0(A)=\rho^{(0)}(A')$,

$$(K\phi)(A)=\phi\left(\tfrac{1}{2}(A''\otimes b_x^{(m+1)})(\alpha_i-\mathbb{1})(\mathbb{1}\otimes b_x^{(m+1)})\right)$$

The equation is $\rho(A)=\rho_0(A)+(K\rho)(A)$

$\iff\quad(1-K)\rho=\rho_0$ Enough to check that $\|K\|<1$.

$\|AB\|_2 \le \|A\| \, \|B\|_2$

Consider $K : \mathcal{L}(\mathcal{A}^{(m+1)}_{\Lambda,x}, \|\cdot\|_2) \to$ self.

Then $\|\phi\| = \sup_{\|A\|_2 = 1} |\phi(A)|$. then $|\phi(A)| \le \|\phi\| \, \|A\|_2$

$$
\begin{aligned}
|(K\phi)(A)| &\le \tfrac{1}{2}\|\phi\| \left\|((A'' \otimes b_x^{(m+1)})(\alpha_{\mathrm{i}} - \mathbb{1})(\mathbb{1}_\Lambda \otimes a_x^{(m+1)})\right\|_2 \\
&\le \tfrac{1}{2}\|\phi\| \|A''\|_2 \|b_x^{(m+1)}\|_2 \, \mathrm{e}^r \frac{2\|\Phi\|_r}{r - 2\|\Phi\|_r} \|a_x^{(m+1)}\|.
\end{aligned}
$$

This uses

$$
\|(\alpha_{\mathrm{i}} - \mathbb{1})(\mathbb{1}_\Lambda \otimes a_x^{(m+1)})\| \le \mathrm{e}^r \frac{2\|\Phi\|_r}{r - 2\|\Phi\|_r} \|a_x^{(m+1)}\|.
$$

Using $\|A''\|_2 \le \sqrt{\frac{N}{2}} \, \|A\|_2$, $\|b_x^{(m+1)}\| = \sqrt{\frac{2}{N}}$, $\|a_x^{(m+1)}\| = 1$, we get

$$
|(K\phi)(A)| \le \|\phi\| \, \|A\|_2 \, \mathrm{e}^r \frac{\|\Phi\|_r}{r - 2\|\Phi\|_r}.
$$

Then $\|K\| < 1$. $\square$

Long-range order in XXZ model

① Peierls argument for Ising

② Kennedy's extension to XXZ with any $J^3 > J^1 = J^2$.

Note: long-range order $\Rightarrow$ not mixing $\Rightarrow$ Gibbs state not extremal

$$\Longrightarrow \boxed{|\,g_\beta^\phi\,| > 1}$$

$N = 2$

Ising hamiltonian:

$$H_\Lambda^{\text{ISING}} = -2 \sum_{xy \in \mathcal{E}(\Lambda)} \left(S_x^{(3)} S_y^{(3)} - \tfrac{1}{4}\right) = -\tfrac{1}{2} \sum_{xy \in \mathcal{E}(\Lambda)} \left(\sigma_x^{(3)} \sigma_y^{(3)} - 1\right).$$

Recall that $\mathcal{H}_\Lambda = \bigotimes_{x \in \Lambda} \mathbb{C}^2 \simeq \ell^2(\{-1,1\}^\Lambda)$.

Notation: $\omega = (\omega_x)_{x \in \Lambda}$, $\omega_x = \pm 1$.

Probability measure on classical configurations:

$$\mathbb{P}_\Lambda(\omega) = \frac{1}{Z_\Lambda} \exp\left(\tfrac{1}{2}\beta \textstyle\sum_{xy \in \mathcal{E}(\Lambda)} (\omega_x \omega_y - 1)\right),$$

Then:

$$\langle \sigma_x^{(3)} \sigma_y^{(3)} \rangle_\Lambda = \mathbb{E}_\Lambda[\omega_x \omega_y].$$

$\langle \sigma_x^{(3)} \sigma_y^{(3)} \rangle =$

$= \langle \sigma_x^3 \sigma_y^3 \rangle - \underbrace{\langle \sigma_x^3 \rangle}_{= 0} \langle \sigma_y^3 \rangle$

Here: $\Lambda = \{-N, \dots, N\}^d$.

Occurrence of long-range order

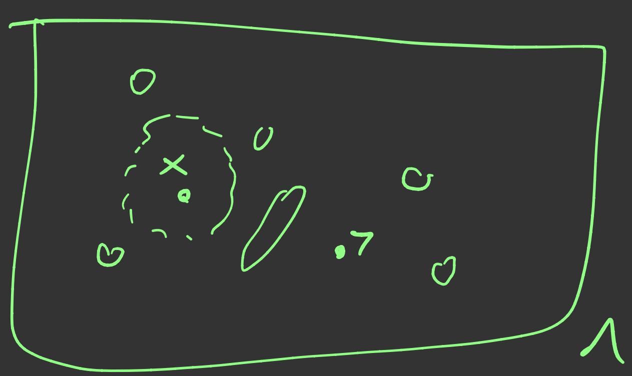
THEOREM 4.2. *Consider the model (4.13) with $d \geq 2$. There exist $\beta_0 < \infty$ and $c(\beta) > 0$ (that depend on d but not on N) such that for $\beta > \beta_0$, we have*

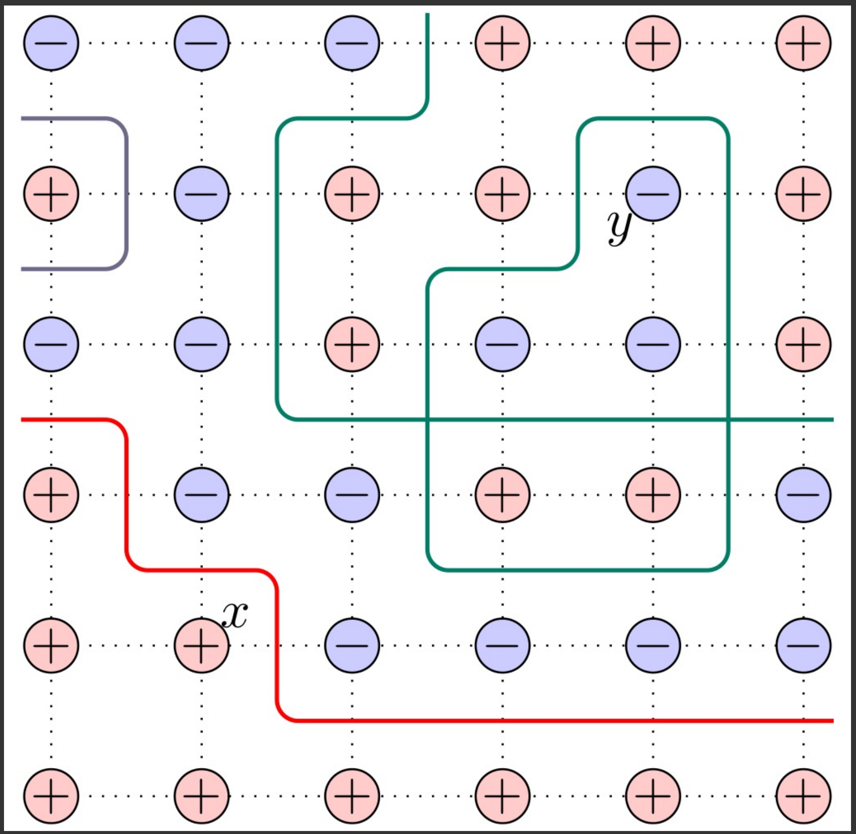
$$\langle \sigma_x^{(3)} \sigma_y^{(3)} \rangle_\Lambda \geq c \tag{4.18}$$

for all $x, y \in \Lambda_N$.

Proof.

Contour representation:





$$Z_{\Lambda,\beta} = \mathrm{Tr}\ \mathrm{e}^{-\beta H_\Lambda^{\mathrm{ISING}}} = 2 \sum_{g \in G_\Lambda} \prod_{\gamma \in g} w(\gamma).$$

where the weights of the contours are $w(\gamma) = \mathrm{e}^{-\beta|\gamma|}$,

$\mathbb{1}_{\omega_x = \omega_y} - \mathbb{1}_{\omega_x \neq \omega_y} = 1 - 2 \cdot \mathbb{1}_{\omega_x \neq \omega_y}$

$\downarrow$

$$\langle \sigma_x^{(3)} \sigma_y^{(3)} \rangle_\Lambda = \mathbb{E}_\Lambda[\omega_x \omega_y] = 1 - 2\mathbb{P}_\Lambda(\omega_x \neq \omega_y).$$

set of sets of disjoint contours

$$\mathbb{P}_\Lambda(\omega_x \neq \omega_y) \leq \frac{2}{Z_{\Lambda,\beta}} \sum_{\substack{g \in G_\Lambda \\ x,y \text{ separated}}} \prod_{\gamma \in g} w(\gamma)$$

$$\leq \frac{2}{Z_{\Lambda,\beta}} \Bigg(\sum_{\gamma_0 : \mathrm{Int}\gamma_0 \ni x} w(\gamma_0) \sum_{g : g \cup \{\gamma_0\} \in G_\Lambda} \prod_{\gamma \in g} w(\gamma) + [\text{same with } y] \Bigg).$$

interior of γ_0

We have:

$$\frac{2}{Z_{\Lambda,\beta}} \sum_{g: g\cup\{\gamma_0\}\in G_\Lambda} \prod_{\gamma\in g} w(\gamma) \le 1.$$

Then

$$\mathbb{P}_\Lambda(\omega_x \neq \omega_y) \le \sum_{\gamma_0 : \mathrm{Int}\gamma_0 \ni x} w(\gamma_0) + \sum_{\gamma_0 : \mathrm{Int}\gamma_0 \ni y} w(\gamma_0).$$

$D_x(\gamma_0)$: length of shortest path between x and γ_0

$$\sum_{\gamma_0 : \mathrm{Int}\gamma_0 \ni x} w(\gamma_0) \le \sum_{\gamma_0 \in \Gamma_\Lambda} \mathrm{e}^{-(\beta - \log\delta)|\gamma_0|}\, \delta^{-D_x(\gamma_0)}.$$

$\forall\ \delta \ge 1$

$D_x(\gamma_0) \le |\gamma_0|$

We now estimate the sum over contours:

$$\sum_{\gamma_0 \in \Gamma_\Lambda} \mathrm{e}^{-(\beta - \log\delta)|\gamma_0|}\, \delta^{-D_x(\gamma_0)} \le \left(\tfrac{2}{\delta - 1}\right)^d \left(c_d^{-1}\, \mathrm{e}^{\beta - \log\delta} - 1\right)^{-1}.$$

We get a bound for the probability that $\omega_x \neq \omega_y$, hence for $\langle S^3_x S^3_y \rangle$:

$$\mathbb{P}_\Lambda(\omega_x \neq \omega_y) \le 2\left(\tfrac{2}{\delta - 1}\right)^d \left(c_d^{-1}\, \mathrm{e}^{\beta - \log\delta} - 1\right)^{-1}.$$

$\square$