# Introduction to Quantum Spin Systems

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CHAPTER 1

# Introduction

Quantum lattice models were introduced in the first half of the 20th century in order to describe the electronic properties of condensed matter systems. The latter physical systems involve many atoms and electrons, which are quantum charged particles interacting via Coulomb forces. Electrons carry a *spin*, a quantum degree of freedom. It is not possible to study this complicated system directly, so physicists have introduced simplified models that retain some of the mechanisms at work.

Quantum spin systems are models that only involve the spins of (some of) the electrons. They allow to shed light on the magnetic properties of the system. All these models assume that atoms form periodic lattices, often taken to be $\mathbb{Z}^d$, and that the relevant electrons are bound to atoms. Then each site hosts a quantum spin, that interact with their nearest-neighbours via two-body interactions.

A special case is the Ising model, which is in fact a classical model. It enjoys special features (such as correlation inequalities) and this has allowed mathematical-physicists to uncover many properties. An excellent account of the Ising model can be found in Chapter 3 of the textbook of Friedli and Velenik [**?**].

Anybody interested in classical lattice systems should give some attention to quantum spin systems. The latter are physically more relevant and they have a rich mathematical structure. One can argue that $d$-dimensional quantum systems are equivalent to $d+1$-dimensional classical systems. Some quantum models have attractive probabilistic graphical representations. These notes review the setting, basic properties, and some probabilistic representations. They are written for readers with knowledge of the Ising model.

The energy of the system is given by a self-adjoint operator on $\mathcal{H}_\Lambda$. It involves interactions between nearest-neighbours and interactions with an external magnetic field. We list here five important hamiltonians; recall that $\mathcal{E}_\Lambda$ denote the set of edges (unordered pairs of nearest-neighbours) of $\Lambda$.

- **Ising**: $H^{\text{Ising}}_{\Lambda,\beta,h} = -\beta \sum_{\{x,y\}\in\mathcal{E}_\Lambda} S^{(3)}_x S^{(3)}_y - h \sum_{x\in\Lambda} S^{(3)}_x.$
- **Quantum Ising**: $H^{\text{qu.Is.}}_{\Lambda,\beta,h} = -\beta \sum_{\{x,y\}\in\mathcal{E}_\Lambda} S^{(1)}_x S^{(1)}_y - h \sum_{x\in\Lambda} S^{(3)}_x.$
- **Quantum XY**: $H^{\text{XY}}_{\Lambda,\beta,h} = -\beta \sum_{\{x,y\}\in\mathcal{E}_\Lambda} \big(S^{(1)}_x S^{(1)}_y + S^{(3)}_x S^{(3)}_y\big) - h \sum_{x\in\Lambda} S^{(3)}_x.$

- **Heisenberg ferromagnet**:
$$H^{\text{Heis.F}}_{\Lambda,\beta,h} = -\beta \sum_{\{x,y\}\in\mathcal{E}_\Lambda} \big( S^{(1)}_x S^{(1)}_y + S^{(2)}_x S^{(2)}_y + S^{(3)}_x S^{(3)}_y \big) - h \sum_{x\in\Lambda} S^{(3)}_x.$$
- **Heisenberg antiferromagnet**:
$$H^{\text{Heis.AF}}_{\Lambda,\beta,h} = +\beta \sum_{\{x,y\}\in\mathcal{E}_\Lambda} \big( S^{(1)}_x S^{(1)}_y + S^{(2)}_x S^{(2)}_y + S^{(3)}_x S^{(3)}_y \big) - h \sum_{x\in\Lambda} S^{(3)}_x.$$

CHAPTER 2

# Setting

## 1. Hilbert spaces, local operators, Gibbs states

We restrict our attention to "spin $\frac12$" systems, where the local spin space is $\mathbb{C}^2$. We also use a setting that is close to that of the Ising model; it is mathematically equivalent to the usual setting involving tensor products of Hilbert spaces.

Let $\Lambda \Subset \mathbb{Z}^d$ denote the domain. Recall that $\Omega_\Lambda = \{-1,1\}^\Lambda$ is the set of classical spin configurations of the Ising model. We consider here the complex Hilbert space

$$\mathcal{H}_\Lambda = \operatorname{span} \Omega_\Lambda \simeq \ell^2(\Omega_\Lambda). \tag{2.1}$$

That is, $\mathcal{H}_\Lambda$ consists of all linear combinations of elements of $\Omega_\Lambda$; $\dim \mathcal{H}_\Lambda = 2^{|\Lambda|}$. We use Dirac's notation: If $\omega \in \Omega_\Lambda$, the corresponding element in $\mathcal{H}_\Lambda$ is denoted $|\omega\rangle$. The inner product between $|\omega\rangle, |\omega'\rangle \in \mathcal{H}_\Lambda$ is written as $\langle\omega|\omega'\rangle$. And if $A$ is an operator on $\mathcal{H}_\Lambda$, the inner product between $|\omega\rangle$ and $A|\omega'\rangle$ is $\langle\omega|A|\omega'\rangle$.

Just as local functions play an important rôle in the Ising model, local operators are needed here. The **support** of an operator $A$ on $\mathcal{H}_\Lambda$ is the smallest subset $X \subset \Lambda$ such that there exists a function $a$ on $\Omega_X \times \Omega_X$ such that we have

$$A|\omega\rangle = \sum_{\substack{\omega'\in\Omega_\Lambda \\ \omega'|_{\Lambda\setminus X} = \omega|_{\Lambda\setminus X}}} a(\omega',\omega)|\omega'\rangle. \tag{2.2}$$

In words, $A|\omega\rangle$ is given by a sum of configurations $\omega'$ that agree with $\omega$ except on the set $X$; the matrix elements $a(\omega',\omega) = \langle\omega'|A|\omega\rangle$ depend on the configurations on the set $X$ only.

We consider hamiltonians that involve pair interactions and a magnetic field. Let $\mathcal{E}_\Lambda$ be the set of edges (nearest-neighbours) of $\Lambda$. Let $T_{x,y}$ be a hermitian operator with support $\{x,y\} \in \mathcal{E}_\Lambda$. The general hamiltonian is defined as

$$H_{\Lambda,\beta,h} = -\beta \sum_{\{x,y\}\in\mathcal{E}_\Lambda} T_{x,y} - h \sum_{x\in\Lambda} S^{(3)}_x. \tag{2.3}$$

Here, $S^{(3)}_x$ is the 3rd spin operator to be defined below; its support is $\{x\}$. This setting includes the most interesting models; this includes in particular the models mentioned in the introduction.

In quantum mechanics, states are given by positive, normalised linear functionals on the space of operators. States are represented by density operators. In order to identify the state that describe equilibrium systems, we invoke thermodynamics. The free energy is equal to the energy minus the temperature times the entropy. With $H$ the hamiltonian and $\beta$ the inverse temperature, the free energy for the state with density operator $\rho$ is

$$F(\rho) = \mathrm{Tr}\, H\rho + \frac{1}{\beta}\mathrm{Tr}\,\rho\log\rho. \tag{2.4}$$

The equilibrium state minimises the free energy. The minimiser of $F$ is the Gibbs state where $\rho = \mathrm{e}^{-\beta H} / \mathrm{Tr}\, \mathrm{e}^{\beta H}$, see Exercise 2.9.

Here we absorb the inverse temperature $\beta$ in the hamiltonian, which also includes an external magnetic field. The finite-volume **Gibbs state** is the positive, normalised, linear map on the space of operators on $\mathcal{H}_\Lambda$ given by

$$\langle A\rangle_{\Lambda,\beta,h} = \frac{1}{Z_{\Lambda,\beta,h}} \mathrm{Tr}_{\mathcal{H}_\Lambda} A\, \mathrm{e}^{-H_{\Lambda,\beta,h}}. \tag{2.5}$$

Here, the normalisation is the **partition function**, equal to

$$Z_{\Lambda,\beta,h} = \mathrm{Tr}_{\mathcal{H}_\Lambda} \mathrm{e}^{-H_{\Lambda,\beta,h}}. \tag{2.6}$$

One easily checks that the map $\langle\cdot\rangle_{\Lambda,\beta,h}$ is a state indeed. We only consider free boundary conditions (or perhaps also periodic boundary conditions); unlike in the classical setting, there are no convenient notions of boundary conditions that allow to define infinite-volume Gibbs states.

## 2. Spin operators

Spin operators are local operators whose support is a single site. They are physically relevant and mathematically convenient, as most symmetries of the hamiltonian are visible. Any operator on $\mathcal{H}_\Lambda$ can be written as sum of products of spin operators.

The spin operators are denoted $\{S_x^{(i)}\}_{x\in\Lambda}^{i=1,2,3}$. Their actions on the basis elements $\{|\omega\rangle\}_{\omega\in\Omega_\Lambda}$ are defined as follows:

$$\begin{aligned} S_x^{(1)}|\omega\rangle &= \tfrac12 |\omega^{(x)}\rangle, \\ S_x^{(2)}|\omega\rangle &= \tfrac12 \mathrm{i}\omega_x |\omega^{(x)}\rangle, \\ S_x^{(3)}|\omega\rangle &= \tfrac12 \omega_x |\omega\rangle. \end{aligned} \tag{2.7}$$

Here, $|\omega^{(x)}$ is the configuration $\omega$ where the spin at $x$ has been flipped. Important properties are stated in Exercises 2.1–2.3, which we encourage the readers to do.

Now that the spin operators have been introduced, the models described in the introduction make sense. Notice that in the case of the Ising hamiltonian all operators are diagonal, so that

$$Z^{\mathrm{Ising}}_{\Lambda,\beta,h} = \mathrm{Tr}\, \mathrm{e}^{-H^{\mathrm{Ising}}_{\Lambda,\beta,h}} = \sum_{\omega\in\Omega_\Lambda} \mathrm{e}^{\frac14\beta\sum_{\{x,y\}}\omega_x\omega_y + \frac12 h\sum_x \omega_x}. \tag{2.8}$$



We recognise the partition function of the usual Ising model with inverse temperature $\frac14\beta$ and magnetic field $\frac12 h$. If we restrict the quantum Gibbs state to diagonal operators, we recover the classical setting.

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EXERCISE 2.1. *Show that $\{S_x^i\}$ satisfy the usual spin relations:*

$$[S_x^{(1)}, S_y^{(2)}] = \mathrm{i}\,\delta_{x,y}\, S_x^{(3)} \qquad \textit{and cyclic permutations of (1,2,3)},$$
$$(S_x^{(1)})^2 + (S_x^{(2)})^2 + (S_x^{(3)})^2 = \tfrac34 \mathbb{1}.$$

EXERCISE 2.2. *Consider $\Lambda = \{x\}$, so that $\mathcal{H}_\Lambda \simeq \mathbb{C}^2$ with basis $\{|+1\rangle, |-1\rangle\}$. Show that the spin operators are given by (one half of) the Pauli matrices, namely*

$$S^{(1)} = \tfrac12\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}, \qquad S^{(2)} = \tfrac12\begin{pmatrix}0 & -\mathrm{i}\\ \mathrm{i} & 0\end{pmatrix}, \qquad S^{(3)} = \tfrac12\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}.$$

EXERCISE 2.3. *Spin operators on $\mathbb{C}^2$ and rotations in $\mathbb{R}^3$. For $\vec a \in \mathbb{R}^3$, let*

$$S^{\vec a} = \vec a \cdot \vec S = a_1 S^{(1)} + a_2 S^{(2)} + a_3 S^{(3)},$$

*where $S^{(i)}$ is the Pauli matrix of the previous exercise. Show that*

(i) $[S^{\vec a}, S^{\vec b}] = \mathrm{i}S^{\vec a\times\vec b}$.
(ii) *Let $R_{\vec a}\vec b$ denote the vector $\vec b$ rotated around $\vec a$ by the angle $\|\vec a\|$. Then*

$$\mathrm{e}^{-\mathrm{i}S^{\vec a}}\, S^{\vec b}\, \mathrm{e}^{\mathrm{i}S^{\vec a}} = S^{R_{\vec a}\vec b}.$$

EXERCISE 2.4. *For each of the following models: Ising, quantum Ising, quantum XY, Heisenberg F and AF (in the form above), show that, for all $x\in\Lambda$, $\beta \geq 0$, and $h\in\mathbb{R}$,*

$$\langle S_x^{(3)}\rangle_{\Lambda,\beta,0} = 0; \qquad \langle S_x^{(1)}\rangle_{\Lambda,\beta,h} = \langle S_x^{(2)}\rangle_{\Lambda,\beta,h} = 0.$$

EXERCISE 2.5. *For any matrices $A$ and $B$, show that*

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathrm{Tr}\, \mathrm{e}^{A+tB} = \mathrm{Tr}\, B\, \mathrm{e}^{A+tB}.$$

*(Duhamel formula may be useful here.)*

EXERCISE 2.6. *Let $h\geq 0$. Show that $\Big\langle \prod_{x\in X} S_x^{(3)} \Big\rangle_{\Lambda,\beta,h} \geq 0$ for all $X\subset\Lambda$.*

EXERCISE 2.7. *Let $\omega$ be a normalised, positive linear functional on the space of bounded operators. Show that $\omega$ is bounded, and that its norm is $\|\omega\| = 1$.*

EXERCISE 2.8. **Riesz representation of states**. *Show that any positive, normalised linear functional $f$ on the space of operators can be written with the help of a density operator as $f(a) = \mathrm{Tr}\,\rho a$.*

EXERCISE 2.9. *Show that the density operator that minimises the free energy (2.4) is the Gibbs operator* $\rho = \mathrm{e}^{-\beta H} / \mathrm{Tr}\, \mathrm{e}^{-\beta H}$.

CHAPTER 3

# Heisenberg model on the complete graph

The most relevant models are defined on lattices $\mathbb{Z}^d$ with $d \geq 2$. But since it is hopelessly difficult to obtain rigorous results, we look at the case of the complete graph. The corresponding Ising model is usually referred to as the "Curie-Weiss model".

## 1. General heuristics

Let us first consider a heuristic that should be relevant for all ferromagnetic systems. We can expect the system to display spontaneous magnetisation that can point in any direction. If we average over the 3rd component of the average magnetisation we should have that, for all $h \in \mathbb{C}$,

$$\begin{aligned} \lim_{\Lambda \Uparrow \mathbb{Z}^d} \Big\langle \mathrm{e}^{\frac{h}{|\Lambda|} \sum_{x \in \Lambda} S_x^{(3)}} \Big\rangle_{\Lambda,\beta,0} &= \int_{\mathbb{S}^2} \mathrm{e}^{hm^* a_3}\, \mathrm{d}\vec{a} \\ &= \frac{1}{4\pi} \int_0^\pi \sin\theta \mathrm{d}\theta \int_0^{2\pi} \mathrm{d}\varphi\, \mathrm{e}^{hm^* \cos\theta} \\ &= \frac{\sinh(hm^*)}{hm^*}. \end{aligned} \tag{3.1}$$

In the first line $\mathrm{d}\vec{a}$ denotes the uniform probability measure on the two-dimensional sphere $\mathbb{S}^2$ and $a_3$ is the 3rd component of the vector $\vec{a}$; $m^*$ is the spontaneous magnetisation, that depends on $\beta$ and on the lattice. If $m^* = 0$, the last term is equal to 1.

An important conjecture is that (3.1) holds when $\Lambda \Uparrow \mathbb{Z}^d$, $d \geq 3$, where $m^* = m^*(\beta)$ is positive if and only if $\beta > \beta_\mathrm{c}$ with $\beta_\mathrm{c} \in (0, \infty)$. Rigorous results about the phase with $m^* > 0$ still elude us. We will prove that (3.1) holds with $m^* = 0$ when $d = 1$ and $d = 2$, and for all dimensions when $\beta$ is small enough. We also prove that (3.1) holds for the model in the complete graph.

## 2. On the complete graph

Let $n \in \mathbb{N}$ be the number of spins. The Hilbert space is $\mathcal{H}_n = \mathrm{span}\{-1, +1\}^n$ with dimension $2^n$. With $\{S_x^{(i)}\}_{x=1,\dots,n}^{i=1,2,3}$ the usual spin operators, we consider the hamiltonian

$$H_n = -\frac{\beta}{n} \sum_{x,y=1}^{n} \big( S_x^{(1)} S_y^{(1)} + S_x^{(2)} S_y^{(2)} + S_x^{(3)} S_x^{(3)} \big). \tag{3.2}$$

Notice that interactions have been divided by $n$, so the total energy scales like the volume of the system. We do not include an external magnetic field here. The Gibbs state is

$$\langle \cdot \rangle_{n,\beta,0} = \frac{1}{\mathrm{Tr}\ \mathrm{e}^{-H_n}} \mathrm{Tr}\ \cdot\ \mathrm{e}^{-H_n}. \tag{3.3}$$

In order to formulate the result, let us introduce the function $\Phi : [0, \frac{1}{2}] \to \mathbb{R}$ by

$$\Phi(s) = \beta s^2 - (\tfrac{1}{2} - s) \log(\tfrac{1}{2} - s) - (\tfrac{1}{2} + s) \log(\tfrac{1}{2} + s). \tag{3.4}$$

A few calculations show that $\Phi(0) = \log 2$, $\Phi(\frac{1}{2}) = \frac{\beta}{4}$, $\Phi'(0) = 0$, $\Phi'(\frac{1}{2}) = -\infty$, $\Phi''(0) = 2\beta - 4$. This function is depicted in Fig. 3.1. Let $m^* = m^*(\beta) \in [0, \frac{1}{2}]$ be the maximiser of $\Phi$. It is positive if and only if $\beta > \beta_{\mathrm{c}} = 2$.

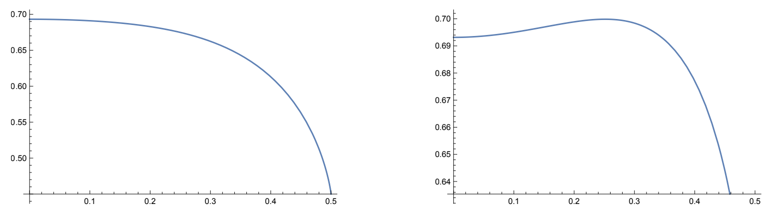


FIGURE 3.1. The function $\Phi_\beta(s)$ defined in Eq. (3.4) with $\beta = 1.8$ (left) and $\beta = 2.2$ (right). The maximiser $m^\star(\beta)$ is positive when $\beta > 2$.

Here is the main result. It shows that a magnetic phase transition take place at $\beta_{\mathrm{c}} = 2$ that is compatible with the heuristic above.

THEOREM 3.1. *Let $m^*$ be the maximiser of the function $\Phi$ defined in Eq. (3.4). Then for every $h \in \mathbb{C}$, we have*

$$\lim_{n\to\infty} \big\langle \mathrm{e}^{\frac{h}{n} \sum_{x=1}^{n} S_x^{(3)}} \big\rangle_{n,\beta,0} = \frac{\sinh(hm^*)}{hm^*}.$$

The rest of the section is devoted to the proof of this theorem. We consider the operators for the total spin,

$$T^{(i)} = \sum_{x=1}^{n} S_x^{(i)}. \tag{3.5}$$



Let $\vec{T}^2 = (T^{(1)})^2 + T^{(2)})^2 + T^{(3)})^2$. Notice that $[(\vec{T})^2, T^{(i)}] = 0$. The hamiltonian is equal to $H_n = -\beta(\vec{T})^2$ because we work on the complete graph. It is possible to get the complete spectrum of $\vec{T}^2$ and $T^{(i)}$.

PROPOSITION 3.2.
(a) *The set of eigenvalues of $T^{(i)}$ is*

$$\mathrm{Eig}(T^{(i)}) = \{-\tfrac{n}{2}, -\tfrac{n}{2}+1, \dots, \tfrac{n}{2}\}.$$

*The multiplicity of $m \in \mathrm{Eig}(T^{(i)})$ is $\binom{n}{\frac{n}{2}+m}$.*

(b) *The set of eigenvalues of $\vec{T}^2$ is*

$$\mathrm{Eig}(\vec{T}^2) = \begin{cases} \{j(j+1) : j = 0, 1, \dots, \frac{n}{2}\} & \text{if } n \text{ is even,} \\ \{j(j+1) : j = \frac{1}{2}, \frac{3}{2}, \dots, \frac{n}{2}\} & \text{if } n \text{ is odd.} \end{cases}$$

(c) *Let $\mathcal{H}_n^{(j)}$ be the eigensubspace where $\vec{T}^2$ has eigenvalue $j(j+1)$, and $\mathcal{H}_n^{(j,m)}$ the eigensubspace where $\vec{T}^2$ has eigenvalue $j(j+1)$ and $T^3$ has eigenvalue $m$. Then*

$$\frac{1}{2j+1} \dim \mathcal{H}_n^{(j)} = \dim \mathcal{H}_n^{(j,m)} = \binom{n}{\frac{n}{2}+j} \frac{2j+1}{\frac{n}{2}+j+1} 1_{|m|\leq j}.$$

PROOF. (a) is easy, since all $T^{(i)}$, $i = 1, 2, 3$, have the same spectrum, and $T^{(3)}|\omega\rangle = \frac{1}{2}\sum_{x=1}^{n} \omega_x$. Notice that $\frac{n}{2} + m$ is the number of $-1$ in the configuration $\omega$.

For (b), let $T^{\pm} = T^{(1)} \pm \mathrm{i} T^{(2)}$. One can check that

$$[T^{(3)}, T^{\pm}] = \pm T^{\pm}, \qquad [T^+, T^-] = 2T^{(3)}. \tag{3.6}$$

Further,

$$T^{\pm} T^{\mp} = \vec{T}^2 - (T^{(3)})^2 \pm T^{(3)}. \tag{3.7}$$

The left side is nonnegative, which implies that

$$|m| \leq j. \tag{3.8}$$

If $|m\rangle$ is eigenvector of $T^{(3)}$ with eigenvalue $m$, then

$$T^{(3)} T^{\pm} |m\rangle = (T^{\pm} T^{(3)} \pm T^{(3)}) |m\rangle = (m \pm 1) |m\rangle. \tag{3.9}$$

Further, if $|m\rangle \in \mathcal{H}_n^{(j)}$,

$$\big\| T^{\pm} |m\rangle \big\|^2 = j(j+1) - m(m \pm 1) \tag{3.10}$$

Then $T^{\pm}|m\rangle$ is eigenvector of $T^{(3)}$ with eigenvalue $m \pm 1$ unless $m = \pm j$, in which case $T^{\pm}|m\rangle$ is zero. In order for the condition (3.8) to be respected, $j$ must be integer if $n$ is even, or half-integer if $n$ is odd. Otherwise the repeated action of $T^{\pm}$ would yield eigenvectors with $|m| > j$. This proves (b).

For (c), let $|j,m,\alpha\rangle$ denote the eigenvectors in $\mathcal{H}_n^{(j,m}$ of eigenvalue $j(j+1)$ for $\vec{T}^2$ and $m$ for $T^{(3)}$; the third index runs from 1 to $\dim \mathcal{H}_n^{(j,m)}$. Since $[\vec{T}^2, T^{\pm}] = 0$, we have that $T^{\pm}|j,m,\alpha\rangle \in \mathcal{H}_n^{(j,m\pm1)}$. Using (3.7), we find that $T^{\pm}|j,m,\alpha\rangle \perp |j,m,\alpha'\rangle$ if $\alpha \neq \alpha'$. It follows that $\dim \mathcal{H}_n^{(j,m)}$ depends on $j$ but not on $m$, so that $\dim \mathcal{H}_n^{(j)} = (2j+1) \dim \mathcal{H}_n^{(j,m)}$. Finally, we have

$$\binom{n}{\frac{n}{2}+m} = \sum_{j=|m|}^{n/2} \dim \mathcal{H}_n^{(j,m)}. \tag{3.11}$$

Then

$$\dim \mathcal{H}_n^{(j,m)} = \binom{n}{\frac{n}{2}+j} - \binom{n}{\frac{n}{2}+j+1}. \tag{3.12}$$

This gives the expression (c). □

Proof of Theorem 3.1. We use Proposition 3.2. We also suppose that $n$ is even for simplicity.

$$\begin{aligned} \operatorname{Tr} \mathrm{e}^{\frac{h}{n}T^{(3)}} \mathrm{e}^{\frac{\beta}{n}\vec{T}^2} &= \sum_{j=0}^{n/2} \sum_{m=-j}^{j} \binom{n}{\frac{n}{2}+j} \frac{2j+1}{\frac{n}{2}+j+1} \mathrm{e}^{\frac{h}{n}m + \frac{\beta}{n}j(j+1)} \\ &= \sum_{j=0}^{n/2} \mathrm{e}^{n\Phi_n(j)} q_{n,j}(h), \end{aligned} \tag{3.13}$$

where

$$\begin{aligned} \Phi_n(j) &= \frac{1}{n}\log\binom{n}{\frac{n}{2}+j} + \frac{1}{n}\log\frac{(2j+1)^2}{\frac{n}{2}+j+1} + \frac{\beta}{n^2}j(j+1), \\ q_{n,j}(h) &= \frac{1}{2j+1}\sum_{m=-j}^{j} \mathrm{e}^{h\frac{m}{n}}. \end{aligned} \tag{3.14}$$

Notice that $q_{n,j}$ is uniformly bounded for $j \leq \frac{n}{2}$, for fixed $h$. Using Stirling's formula we get

$$\Phi_n(j) = \Phi(\tfrac{j}{n}) + o(1). \tag{3.15}$$



Then by Laplace's principle,

$$\begin{aligned} \lim_{n\to\infty} \left\langle \mathrm{e}^{\frac{h}{n}T^{(3)}} \right\rangle_{n,\beta,0} &= \lim_{n\to\infty} \frac{\sum_{j=0}^{n/2} \mathrm{e}^{n\Phi_n(j)} q_{n,j}(h)}{\sum_{j=0}^{n/2} \mathrm{e}^{n\Phi_n(j)}} \\ &= \lim_{n\to\infty} \frac{1}{2m^*n+1} \sum_{m=-m^*n}^{m^*n} \mathrm{e}^{\frac{h}{n}m} \\ &= \frac{1}{2m^*}\int_{-m^*}^{m^*} \mathrm{e}^{hs}\, \mathrm{d}s \\ &= \frac{\sinh(hm^*)}{hm^*}. \end{aligned} \tag{3.16}$$

□

CHAPTER 4

# Infinite-volume limits

Recall that a sequence of finite domains $(\Lambda_n)_{n\geq 1}$ is a **van Hove sequence** if the domains approach $\mathbb{Z}^d$ in such a way that the fraction of sites at the boundary goes to 0. Precisely, the sequence satisfies

(i) $\Lambda_{n+1} \supset \Lambda_n$ for all $n$;
(ii) $\cup_{n\geq 1} \Lambda_n = \mathbb{Z}^d$;
(iii) $\frac{|\partial_{\rm in} \Lambda_n|}{|\Lambda_n|} \to 0$ as $n \to \infty$.

Here, the inner boundary is $\partial_{\rm in}\Lambda = \{x \in \Lambda : \exists y \in \Lambda^{\rm c} \text{ with } \{x,y\} \in \mathcal{E}_\Lambda\}$.

## 1. Pressure and magnetisation

The pressure is defined as the logarithm of the partition function divided by the volume, as in the classical case. It is also convex in $\beta$ and $h$; the proof is more difficult and it involves the Hölder and Golden-Thompson inequalities; to prove the latter, we need the Trotter product formula.

The **pressure** is

$$\psi_\Lambda(\beta, h) = \frac{1}{|\Lambda|} \log Z_{\Lambda,\beta,h}. \tag{4.1}$$

Theorem 4.1.

(a) *The pressure $\psi_\Lambda(\beta,h)$ is jointly convex in $(\beta,h)$.*
(b) *Assume in addition that $[T_{x,y}, S_x^{(i)} + S_y^{(i)}] = 0$ for either $i=1$ or $i=2$ (or both), and all $\{x,y\} \in \mathcal{E}_\Lambda$. Then $\psi_\Lambda(\beta,h)$ is even in $h$.*

The claim (a) about convexity is more elegant to prove in much greater generality; see Theorem 4.2 below. For the claim (b), we cannot invoke a "spin flip" symmetry in the quantum case, but we can use spin rotations. Let $U_\Lambda = \prod_{x\in\Lambda} {\rm e}^{{\rm i}\pi S_x^{(i)}}$ be the unitary that rotates the spin operators around the $i$th axis by angle $\pi$ (see Exercise 2.3). We have $U_\Lambda^{-1} T_{x,y} U_\Lambda = T_{x,y}$ and $U_\Lambda^{-1} S_x^{(3)} U_\Lambda = -S_x^{(3)}$, so that

$$Z_{\Lambda,\beta,h} = {\rm Tr}\, {\rm e}^{-H_{\Lambda,\beta,h}} = {\rm Tr}\, U_\Lambda^{-1}\, {\rm e}^{-H_{\Lambda,\beta,h}}\, U_\Lambda = {\rm Tr}\, {\rm e}^{-H_{\Lambda,\beta,-h}} = Z_{\Lambda,\beta,-h}. \tag{4.2}$$

Then $\psi_\Lambda(\beta,h) = \psi_\Lambda(\beta,-h)$.

THEOREM 4.2 (Convexity of the abstract pressure). *The function*

$$f(A) = \log \mathrm{Tr}\ \mathrm{e}^{A}$$

*is a convex function on the space of hermitian matrices.*

PROOF. We use the Golden-Thompson inequality (Proposition A.7) and then the Hölder inequality (Proposition A.1).

$$\begin{aligned} f(sA + (1-s)B) &= \log \mathrm{Tr}\ \mathrm{e}^{sA+(1-s)B} \\ &\leq \log \mathrm{Tr}\ \mathrm{e}^{sA}\ \mathrm{e}^{(1-s)B} \\ &\leq \log\Big[\Big(\mathrm{Tr}\ \big(\mathrm{e}^{sA}\big)^{\frac{1}{s}}\Big)^{s}\Big(\mathrm{Tr}\ \big(\mathrm{e}^{(1-s)B}\big)^{\frac{1}{1-s}}\Big)^{1-s}\Big] \\ &= sf(A) + (1-s)f(B). \end{aligned} \tag{4.3}$$

□

We now turn to the thermodynamic limit (i.e. infinite-volume limit) of the pressure. Recall that $\Lambda \Uparrow \mathbb{Z}^d$ denote the limit in the sense of van Hove.

THEOREM 4.3. *There exists a function $\psi(\beta, h)$, that is convex in $(\beta, h)$ and even in $h$, such that*

$$\psi(\beta, h) = \lim_{n\to\infty} \psi_{\Lambda_n}(\beta, h)$$

*along all sequences of domains such that $\Lambda_n \Uparrow \mathbb{Z}^d$.*

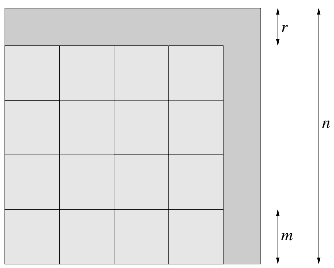


FIGURE 4.1. The large box of size $n$ is decomposed in $k^d$ boxes of size $m$; there are no more than $drn^{d-1}$ remaining sites in the darker area.



PARTIAL PROOF. We only consider the sequence $(\Lambda_n)$ of boxes of size $n$. We use a subadditive argument. We can assume that $T_{x,y} \geq 0$, adding a constant if necessary. Notice that the inequality $\mathrm{Tr}\ \mathrm{e}^{A+B} \geq \mathrm{Tr}\ \mathrm{e}^{A}$ holds for all self-adjoint operators $A, B$ with $B \geq 0$. (This follows e.g. from the minimax principle, or from Klein's inequality, Proposition A.8.)

Let $m, n, k, r$ be integers such that $n = km + r$ and $0 \leq r < m$. The box $\Lambda_n$ is the disjoint union of $k^d$ boxes of size $m$, and of some remaining sites (fewer than $drn^{d-1}$); see Figure 4.1 for an illustration. We get an inequality for the partition function in $\Lambda_n$ by dismissing all $T_{x,y}$ where $\{x, y\}$ are not inside a single box of size $m$. The boxes $\Lambda_m$ become independent, and

$$\begin{aligned} Z_{\Lambda_n,\beta,h} &= \mathrm{Tr}\ \exp\Big(\beta \sum_{\{x,y\}\in\mathcal{E}_{\Lambda_n}} h_{x,y} + h \sum_{x\in\Lambda_n} S_x^{(3)}\Big) \\ &\geq \Big[\mathrm{Tr}_{\mathcal{H}_{\Lambda_m}} \exp\Big(\beta \sum_{\{x,y\}\in\mathcal{E}_{\Lambda_m}} h_{x,y} + h \sum_{x\in\Lambda_m} S_x^{(3)}\Big)\Big]^{k^d} \\ &= [Z_{\Lambda_m,\beta,h}]^{k^d}. \end{aligned} \tag{4.4}$$

We have neglected the contribution of $\mathrm{e}^{hS_x^3}$ for $x$ outside the boxes $\Lambda_m$, which is possible because their traces are greater than 1. It is not hard to check that

$$|\mathcal{E}_{\Lambda_n}| \leq k^d |\mathcal{E}_{\Lambda_m}| + k^d dm^{d-1} + d^2 rn^{d-1}. \tag{4.5}$$

We then obtain a subbaditive relation for the free energy:

$$\psi_{\Lambda_n}(\beta, h) \geq \frac{(km)^d}{n^d} \psi_{\Lambda_m}(\beta, h). \tag{4.6}$$

Then, since $\frac{km}{n} \to 1$ as $n \to \infty$,

$$\liminf_{n\to\infty} \psi_{\Lambda_n}(\beta, h) \geq \psi_{\Lambda_m}(\beta, h). \tag{4.7}$$

Taking the lim sup over $m$ in the right side, we see that it is smaller or equal to the lim inf, and so the limit necessarily exists.

The extension to general van Hove sequences can be done like in Friedli-Velenik. □

COROLLARY 4.4 (Thermodynamic limit with periodic boundary conditions). *Let $(\Lambda_n^{\mathrm{per}})$ be the sequence of cubes in $\mathbb{Z}^d$ of size $n$ with periodic boundary conditions and nearest-neighbor edges. Then $(\psi_{\Lambda_n^{\mathrm{per}}}(\beta, h))_{n\geq 1}$ converges pointwise to the same function $\psi(\beta, h)$ as in Theorem 4.3, uniformly on compact sets.*

This follows from $|\psi_{\Lambda_n^{\mathrm{per}}}(\beta, h) - \psi_{\Lambda_n}(\beta, h)| \leq \frac{\beta d}{n} \|T_{x,y}\|$, which is not too hard to prove, and Theorem 4.3.

The finite-volume **magnetisation** $m_\Lambda(\beta,h)$ is defined as

$$m_\Lambda(\beta,h) = \frac{\partial}{\partial h}\psi_\Lambda(\beta,h) = \Big\langle \frac{1}{|\Lambda|}\sum_{x\in\Lambda} S_x^{(3)} \Big\rangle_{\Lambda,\beta,h}. \tag{4.8}$$

In order to verify the last identity, you may want to use Exercise 2.5.

THEOREM 4.5. *First, assume that $\psi(\beta,h)$ is differentiable in $h$ at $(\beta,h)$; then*

(a) *The limit $m(\beta,h) = \lim_{\Lambda\Uparrow\mathbb{Z}^d} m_\Lambda(\beta,h)$ exists.*

(b) *We have $m(\beta,h) = \frac{\partial}{\partial h}\psi(\beta,h)$.*

*Second, without assuming that $\psi(\beta,h)$ is differentiable in $h$ at $(\beta,h)$, we have*

(c) $\displaystyle \frac{\partial}{\partial h^+}\psi(\beta,h) = \lim_{h'\to h^+} \liminf_{\Lambda\Uparrow\mathbb{Z}^d} \Big\langle \frac{1}{|\Lambda|}\sum_{x\in\Lambda} S_x^{(3)} \Big\rangle_{\Lambda,\beta,h'}.$

Part (c) also holds with lim sup instead of lim inf. We have proved parts (a) and (b) of this theorem in the case of the Ising model. The proof actually extends to the quantum case without modifications.

PROOF. We use the fact that

$$\begin{aligned} \limsup_i \big(\inf_j a_{ij}\big) &\leq \inf_j \big(\limsup_i a_{ij}\big), \\ \liminf_i \big(\sup_j\big) &\geq \sup_j \big(\liminf_i a_{ij}\big), \end{aligned} \tag{4.9}$$

and the following expressions for left- and right-derivatives of convex functions:

$$\begin{aligned} \frac{\mathrm{d}f}{\mathrm{d}h^-}(h) &= \sup_{s>0} \frac{f(h)-f(h-s)}{s}, \\ \frac{\mathrm{d}f}{\mathrm{d}h^+}(h) &= \inf_{s>0} \frac{f(h+s)-f(h)}{s}. \end{aligned} \tag{4.10}$$



Since $\psi$ is convex, we have

$$\begin{aligned} \frac{\partial\psi}{\partial h^-}(\beta,h) &= \sup_{s>0} \liminf_{\Lambda\Uparrow\mathbb{Z}^d} \frac{\psi_\Lambda(\beta,h)-\psi_\Lambda(\beta,h-s)}{s} \\ &\leq \liminf_{\Lambda\Uparrow\mathbb{Z}^d} \frac{\partial\psi_\Lambda}{\partial h}(\beta,h) \\ &\leq \limsup_{\Lambda\Uparrow\mathbb{Z}^d} \frac{\partial\psi_\Lambda}{\partial h}(\beta,h) \\ &= \limsup_{\Lambda\Uparrow\mathbb{Z}^d} \inf_{s>0} \frac{\psi_\Lambda(\beta,h+s)-\psi_\Lambda(\beta,h)}{s} \\ &= \frac{\partial\psi}{\partial h^+}(\beta,h). \end{aligned} \tag{4.11}$$

But $\psi$ is differentiable, so that inequalities are identities and we get

$$\frac{\partial\psi}{\partial h}(\beta,h) = \liminf_{\Lambda\Uparrow\mathbb{Z}^d} \frac{\partial\psi_\Lambda}{\partial h}(\beta,h) = \limsup_{\Lambda\Uparrow\mathbb{Z}^d} \frac{\partial\psi_\Lambda}{\partial h}(\beta,h). \tag{4.12}$$

Since $\frac{\partial\psi_\Lambda}{\partial h}(\beta,h) = m_\Lambda(\beta,h)$, this shows that the infinite volume limit of the latter exists, and is equal to the derivative of $\psi$. This proves (a) and (b).

For (c), let $h_n \to h^+$ such that $\psi$ is differentiable in $h$ at $(\beta,h_n)$. We have just proved that $\frac{\partial}{\partial h^+}\psi(\beta,h_n) = \lim_\Lambda m_\Lambda(\beta,h_n)$. Since right-derivatives of convex functions are right-continuous, and since $m_\Lambda$ is nondecreasing in $h$, we get the result. $\square$

## 2. Gibbs states

TBA

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EXERCISE 4.1. *Let $P_\varepsilon$ be the projector onto the subspace of $\mathcal{H}_\Lambda$ spanned by configurations $\omega$ such that $\frac{1}{|\Lambda|}\sum_x \omega_x \notin (-\varepsilon,\varepsilon)$. Show that if $\psi(\beta,h)$ is differentiable in $h$ at $(\beta,0)$, then*

$$\langle P_\varepsilon\rangle_{\Lambda,\beta,0} \leq \mathrm{e}^{-c|\Lambda|}$$

*for some $c>0$ that is uniform in $\Lambda$. (Hint: Show that a Chernov inequality holds.)*

CHAPTER 5

# High temperature expansions

We prove that correlations have exponential decay at high temperatures, that is, when $\beta$ is small. We also show that there is no spontaneous magnetisation. We consider the hamiltonian defined in Eq. (2.3) with $h = 0$. We assume that for all $k \geq$, all $\{x_1, y_1\}, \dots, \{x_k, y_k\} \in \mathcal{E}_\Lambda$, and all $x \in \Lambda$, we have

$$\operatorname{Tr}\Big(S_x^{(3)} \prod_{i=1}^{k} T_{x_i,y_i}\Big) = 0. \tag{5.1}$$

This property is usually easy to verify in a given model by using spin rotation symmetries. We also assume that $\|T_{x,y}\| \leq 1$ for all $\{x, y\} \in \mathcal{E}_\Lambda$.

THEOREM 5.1. *Under the above assumptions, there exist constants $\beta_0, c >$ 0 such that for all $\beta < \beta_0$ and all $\Lambda \Subset \mathbb{Z}^d$, we have for all $x, y \in \Lambda$,*

$$\big|\big\langle S_x^{(3)} S_y^{(3)} \big\rangle_{\Lambda,\beta,0}\big| \leq \mathrm{e}^{-c\|x-y\|_1}.$$

The decay is given in terms of the $\ell_1$ distance $\|x\|_1 = \sum_{i=1}^d |x_i|$ for convenience. It is also possible to consider the case $h \neq 0$ where exponential decay holds for the *truncated* correlation function, namely

$$\big|\big\langle S_x^{(3)} S_y^{(3)} \big\rangle_{\Lambda,\beta,h} - \big\langle S_x^{(3)} \big\rangle_{\Lambda,\beta,h} \big\langle S_y^{(3)} \big\rangle_{\Lambda,\beta,h}\big| \leq \mathrm{e}^{-c\|x-y\|}. \tag{5.2}$$

The proof can be made using the method of cluster expansions. The case $h = 0$ that we prove here is simpler and is similar to the case of the Ising model.

PROOF OF THEOREM 5.1. We actually make an extra assumption, namely that

$$\langle \omega | T_{x,y} | \omega' \rangle \geq 0 \tag{5.3}$$

for all $\omega, \omega' \in \Omega_\Lambda$ and all $\{x, y\} \in \mathcal{E}_\Lambda$. This assumption can be seen to hold in all the models mentioned in the introduction (adding a harmless constant for diagonal elements if necessary). Without the assumption, we would need to use cluster expansions.

We start by expanding the exponential in Taylor series.

$$\begin{aligned}\mathrm{Tr}\, S_0^{(3)} S_z^{(3)}\, \mathrm{e}^{\beta \sum_{\{x,y\}\in\mathcal{E}_\Lambda} T_{x,y}} &= \mathrm{Tr}\, S_0^{(3)} S_z^{(3)} \sum_{k\geq 0} \frac{\beta^k}{k!} \Big( \sum_{\{x,y\}\in\mathcal{E}_\Lambda} T_{x,y}\Big)^k \\ &= \sum_{k\geq 0}\frac{\beta^k}{k!}\sum_{\{x_1,y_1\},\dots,\{x_k,y_k\}} \mathrm{Tr}\, S_0^{(3)} S_z^{(3)} \prod_{i=1}^k T_{x_i,y_i}.\end{aligned} \tag{5.4}$$

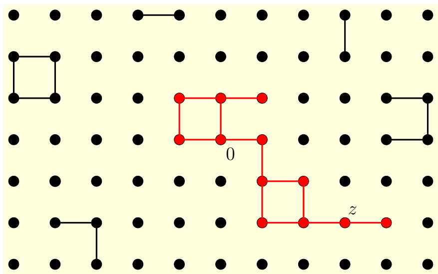



FIGURE 5.1. Illustration for the high temperature expansion. The lines denote the edges that appear in the sum (a given edge can appear multiple times). The sites of $C_0$ that are connected to the origin are marked in red.

Given $\{x_1,y_1\},\dots,\{x_k,y_k\}$, let $C_0$ denote the set of sites that are connected to the origin, see Fig. 5.1 for an illustration. Since the operators $T_{x,y}$ commute when their support $\{x,y\}$ do not overlap, the trace factorises. If $z \notin C_0$, we have

$$\begin{aligned}\mathrm{Tr}_{\mathcal{H}_\Lambda} S_0^{(3)} S_z^{(3)} \prod_{i=1}^k T_{x_i,y_i} = &\Big( \mathrm{Tr}_{\mathcal{H}_{C_0}} S_0^{(3)} \prod_{i:\{x_i,y_i\}\subset C_0} T_{x_i,y_i}\Big) \\ &\Big( \mathrm{Tr}_{\mathcal{H}_{\Lambda\setminus C_0}} S_z^{(3)} \prod_{i:\{x_i,y_i\}\subset \Lambda\setminus C_0} T_{x_i,y_i}\Big),\end{aligned} \tag{5.5}$$

which is equal to zero because of (5.1). If $z \in C_0$, the contribution may be positive. We sum over edges by first choosing the set $C_0$, then summing over $\ell$ edges inside and $m$ edges outside of $C_0$. There is a factor $\binom{\ell+m}{\ell}$ which accounts for all the ways of interweaving the two groups of edges. We get



$$\langle S_0^{(3)} S_z^{(3)} \rangle_{\Lambda,\beta,0} = \sum_{C_0 \ni 0,z} \sum_{\ell \geq |C_0|-1} \frac{\beta^\ell}{\ell!} \sum_{\substack{\{x_1,y_1\},\dots,\{x_\ell,y_\ell\}\subset C_0 \\ C_0 \text{ is connected}}} \mathrm{Tr}_{\mathcal{H}_{C_0}} S_0^{(3)} S_z^{(3)} \prod_{i=1}^\ell T_{x_i,y_i} \cdot \frac{\sum_{m\geq 0} \frac{\beta^m}{m!} \sum_{\{x_1,y_1\},\dots,\{x_\ell,y_\ell\}\subset \Lambda\setminus C_0} \mathrm{Tr}_{\mathcal{H}_{\Lambda\setminus C_0}} \prod_{i=1}^m T_{x_i,y_i}}{\sum_{k\geq 0}\frac{\beta^k}{k!} \sum_{\{x_1,y_1\},\dots,\{x_\ell,y_\ell\}\subset\Lambda} \mathrm{Tr}_{\mathcal{H}_\Lambda} \prod_{i=1}^k T_{x_i,y_i}}. \tag{5.6}$$

The first sum is over connected subsets $C_0$ that contain 0 and $z$; the sum over edges $\{x_1,y_1\},\dots,\{x_\ell,y_\ell\}$ is subject to the constraint that the set of sites that are connected to the origin, is precisely equal to $C_0$. This is only possible if $\ell \geq |C_0|-1$. All terms are positive because of (5.3). The fraction in the last line is less than $2^{-|C_0|}$.

Since $C_0$ connects the sites 0 and $z$, its size is at least $\|z\|_1 + 1$. We now use the convenient trick of inserting the decay we want, with a correction to make it mathematically sound. Namely, for any constants $c, c'$,

$$\begin{aligned}\langle S_0^{(3)} S_z^{(3)} \rangle_{\Lambda,\beta,0} \leq \mathrm{e}^{-c\|z\|_1}\, 2^{-|C_0|} \sum_{C_0\ni 0,z} \mathrm{e}^{-c'(|C_0|-1)} \sum_{\ell\geq|C_0|-1} \frac{(\beta\, \mathrm{e}^{c+c'})^\ell}{\ell!} \\ \sum_{\{x_1,y_1\},\dots,\{x_\ell,y_\ell\}\subset C_0} \mathrm{Tr}_{\mathcal{H}_{C_0}} S_0^{(3)} S_z^{(3)} \prod_{i=1}^\ell T_{x_i,y_i}.\end{aligned} \tag{5.7}$$

We can bound the trace by $\frac14 2^{|C_0|}$. The sum over $\ell$ edges in $C_0$ is less than $(2d|C_0|)^\ell$ so that the sum over $\ell$ gives $\mathrm{e}^{2d\beta\, \mathrm{e}^{c+c'} |C_0|}$. We get

$$\langle S_0^{(3)} S_z^{(3)} \rangle_{\Lambda,\beta,0} \leq \mathrm{e}^{-c\|z\|_1}\, \tfrac14\, \mathrm{e}^{2d\beta\, \mathrm{e}^{c+c'}} \sum_{C_0\ni 0} \mathrm{e}^{-(c'-2d\beta\, \mathrm{e}^{c+c'})(|C_0|-1)}. \tag{5.8}$$

The sum is still over connected sets that contain the origin. There are less than $(2d)^{2(n-1)}$ sets of cardinality $n$ (Exercise 5.1). We can choose $c > 0$, $c'$ large enough, and $\beta$ small enough so that

$$\tfrac14\, \mathrm{e}^{2d\beta\, \mathrm{e}^{c+c'}} \sum_{n\geq 1} (2d)^{2(n-1)}\, \mathrm{e}^{-(c'-2d\beta\, \mathrm{e}^{c+c'})(n-1)} \leq 1. \tag{5.9}$$

This gives the bound stated in the theorem. □

EXERCISE 5.1. *Prove that the number of connected sets of size $n$ in $\mathbb{Z}^d$ that contains a given site, is less than $(2d)^{2(n-1)}$.*

CHAPTER 6

# No spontaneous magnetisation in two dimensions

In Physics, the "Mermin-Wagner theorem" is a principle that states that continuous symmetries cannot be broken in two dimensions. It is backed by several rigorous results (actual theorems...) in specific models. The standard results deal with the fact that any infinite-volume Gibbs states must respect the continuous symmetry of the model.

Here we show that the two-point correlation functions have at least power-law decay, for all temperatures. Decay of correlations was first proposed by Fisher-Jasnow (1971), Bonato-Fernando Perez-Klein (1982), and Ito (1982); these results use the Fourier transform and they only apply to the regular lattice $\mathbb{Z}^2$. A more general result, similar to Theorem 6.2, was proposed in Koma-Tasaki (1992) for the Hubbard model. The present proof is a bit simpler and can be found in Fröhlich-Ueltschi (2015).

We have in mind the lattice $\mathbb{Z}^2$ but we can consider a more general setting with the graph $(\Lambda, \mathcal{E}_\Lambda)$, where $\Lambda$ is a finite set of vertices and $\mathcal{E}_\Lambda$ is the set of edges. We assume that the graph is two-dimensional in the sense that we assume that there exists a constant $K$ such that for all $\ell \in \mathbb{N}$,

$$\#\big\{ \{x,y\} \in \mathcal{E}_\Lambda : d(0,x) = \ell \text{ and } d(0,y) = \ell + 1 \big\} \leq K\ell. \tag{6.1}$$

Here, $d(x,y)$ denotes the graph distance, i.e. the length of the shortest path that connects the vertices $x$ and $y$.

The decay of correlations is measured by the following expression:

$$\xi_\beta(x) = \sup_{\substack{(\phi_y) \in \mathbb{R}^\Lambda \\ \phi_x = 0}} \Big[ \phi_0 - \tfrac12 \beta \sum_{\{y,z\} \in \mathcal{E}_\Lambda} \big( \cosh(\phi_y - \phi_z) - 1 \big) \Big]. \tag{6.2}$$

The solution of this variational problem is essentially a discrete harmonic function. We first obtain a lower bound that applies to all graphs that satisfy the condition (6.1).

LEMMA 6.1. *Assume that the condition (6.1) is satisfied. Then there exists $C = C(\beta, K)$, which does not depend on $x$, such that*

$$\xi_\beta(x) \geq \frac{1}{2\beta K} \log\big( d(0,x) + 1 \big) - C.$$

Proof. With $c$ to be chosen later, let

$$\phi_y = \begin{cases} c \log \frac{d(0,x)+1}{d(0,y)+1} & \text{if } d(0,y) \leq d(0,x), \\ 0 & \text{otherwise.} \end{cases} \tag{6.3}$$

Then

$$\xi_\beta(x) \geq c \log(d(0,x)+1) - \tfrac12 \beta K \sum_{\ell=0}^{d(0,x)-1} \big( \cosh(c \log \tfrac{\ell+2}{\ell+1}) - 1 \big) \ell. \tag{6.4}$$

From the Taylor expansions of the logarithm and of the hyperbolic cosine, there exist $C, C'$ such that

$$\begin{aligned} \xi_\beta(x) &\geq c \log(d(0,x)+1) - \tfrac12 \beta K c^2 \sum_{\ell=1}^{d(0,x)} \frac{1}{\ell} - C' \\ &\geq \big[ c - \tfrac12 \beta K c^2 \big] \log(d(0,x)+1) - C. \end{aligned} \tag{6.5}$$

The optimal choice is $c = (\beta K)^{-1}$. □

We now introduce a family of hamiltonians that include many interesting models, such as Heisenberg and XY. The Hilbert space is $\mathcal{H}_\Lambda = \text{span}\Omega_\Lambda$ with $\Omega_\Lambda = \{-1,+1\}^\Lambda$ as usual. The hamiltonian is

$$H_\Lambda = -\beta \sum_{\{x,y\} \in \mathcal{E}_\Lambda} \big( S_x^{(1)} S_y^{(1)} + S_x^{(2)} S_y^{(2)} + D_{x,y} \big), \tag{6.6}$$

where $D_{x,y}$ is diagonal in the basis of classical configurations, namely $D_{x,y}|\omega\rangle = D(\omega_x,\omega_y)|\omega\rangle$, for some function $D(\omega_x,\omega_y) \in \mathbb{R}$. Typical examples are $D_{x,y} = S_x^{(3)} S_y^{(3)}$, or $D_{x,y} = \frac{h}{\beta}(S_x^{(3)} + S_y^{(3)})$ to describe a magnetic field. We consider correlations in the spin directions 1 and 2 and the upper bound turns out to be independent of $D_{x,y}$. The model is invariant under spin rotations around the 3rd direction of spins.

Theorem 6.2. *Consider the Gibbs state with hamiltonian 6.6. Then, for $i = 1,2$, we have*

$$\big| \langle S_0^i S_x^i \rangle_{\Lambda,\beta} \big| \leq \tfrac12 \, \mathrm{e}^{-\xi_\beta(x)}.$$

In the case of 2D-like graphs, we can use Lemma 6.1 and we obtain power-law decay with a power greater than $(\beta K)^{-1}$, namely $\big|\langle S_0^i S_x^i \rangle_{\Lambda,\beta}\big| \leq (\|x\|_1 + 1)^{-c/\beta}$.

Theorem 6.2 is most relevant at low temperatures, since we have proved in the previous chapter that exponential decay holds very generally. Some models undergo a *Berezinskii–Kosterlitz–Thouless transition* with the decay of correlations going from exponential to power-law when the temperature is lowered. This is expected in the two-dimensional quantum XY model, and the power is expected to be proportional to $\beta^{-1}$ for large $\beta$. Proving a lower bound is much more difficult, Fröhlich and Spencer managed to do it for the *classical* XY model. As for the Heisenberg model, it is expected that correlations decay exponentially for all temperatures in two dimensions, but there exist no proofs (not even for the classical model) and some theoretical physicists even doubt it.



Proof of Theorem 6.2. We use the method of complex rotations. Let

$$S_y^\pm = S_y^{(1)} \pm \mathrm{i} S_y^{(2)}. \tag{6.7}$$

One can check that for any $a \in \mathbb{C}$, we have

$$\mathrm{e}^{a S_y^{(3)}} \, S_y^\pm \, \mathrm{e}^{-a S_y^{(3)}} = \mathrm{e}^{\pm a} \, S_y^\pm. \tag{6.8}$$

The hamiltonian (6.6) can be rewritten as

$$H_\Lambda = -\beta \sum_{\{y,z\} \in \mathcal{E}_\Lambda} \big( S_y^+ S_z^- + S_y^- S_z^+ + D_{y,z} \big) \tag{6.9}$$

Given numbers $\phi_y$, let

$$A = \prod_{y \in \Lambda} \mathrm{e}^{\phi_y S_y^3}. \tag{6.10}$$

Then

$$\operatorname{Tr} S_0^+ S_x^- \, \mathrm{e}^{-H_\Lambda} = \operatorname{Tr} A S_0^+ S_x^- A^{-1} \, \mathrm{e}^{-A H_\Lambda A^{-1}}. \tag{6.11}$$

We now compute the rotated hamiltonian.

$$\begin{aligned} A H_\Lambda A^{-1} &= -\beta \sum_{\{y,z\} \in \mathcal{E}_\Lambda} \big( \mathrm{e}^{\phi_y - \phi_z} \, S_y^+ S_z^- + \mathrm{e}^{-\phi_y + \phi_z} \, S_y^- S_z^+ + D_{y,z} \big) \\ &= H_\Lambda - \beta \sum_{\{y,z\} \in \mathcal{E}_\Lambda} \big( \cosh(\phi_y - \phi_z) - 1 \big) \, (S_y^+ S_z^- + S_y^- S_z^+) \\ &\qquad - \beta \sum_{\{y,z\} \in \mathcal{E}_\Lambda} \sinh(\phi_y - \phi_z) \, (S_y^+ S_z^- - S_y^- S_z^+) \\ &\equiv H_\Lambda + B + C. \end{aligned} \tag{6.12}$$

Notice that $B^* = B$ and $C^* = -C$. We obtain

$$\operatorname{Tr} S_0^+ S_x^- \, \mathrm{e}^{-H_\Lambda} = \mathrm{e}^{\phi_0 - \phi_x} \operatorname{Tr} S_0^+ S_x^- \, \mathrm{e}^{-H_\Lambda - B - C}. \tag{6.13}$$

We now estimate the trace in the right side using the Trotter product formula and the Hölder inequality for traces. Recall that $\|B\|_s = (\operatorname{Tr} |B|^s)^{1/s}$, with $\|B\|_\infty = \|B\|$ being the usual operator norm.

$$\begin{aligned} \operatorname{Tr} S_0^+ S_x^- \, \mathrm{e}^{-H_\Lambda - B - C} &= \lim_{N \to \infty} \operatorname{Tr} S_0^+ S_x^- \Big( \mathrm{e}^{-\frac1N H_\Lambda} \, \mathrm{e}^{-\frac1N B} \, \mathrm{e}^{-\frac1N C} \Big)^N \\ &\leq \lim_{N \to \infty} \|S_0^+ S_x^-\|_\infty \, \big\| \mathrm{e}^{-\frac1N H_\Lambda} \big\|_N^N \, \big\| \mathrm{e}^{-\frac1N B} \big\|_\infty^N \, \big\| \mathrm{e}^{-\frac1N C} \big\|_\infty^N. \end{aligned} \tag{6.14}$$

Observe now that $\| \mathrm{e}^{-\frac{1}{N} H_\Lambda} \|_N^N = Z_{\Lambda,\beta}$, $\| \mathrm{e}^{-\frac{1}{N} B} \|^N \leq \mathrm{e}^{\beta \|B\|}$, and $\| \mathrm{e}^{-\frac{1}{N} C} \| = 1$. We also have $\|S_0^+ S_x^-\| = \frac{1}{2}$. The theorem then follows from

$$\|B\| \leq \tfrac{1}{2}\beta \sum_{\{y,z\} \in \mathcal{E}_\Lambda} \big( \cosh(\phi_y - \phi_z) - 1 \big). \tag{6.15}$$

□

CHAPTER 7

# The quantum Ising model

The quantum Ising model is a quantum system that turns out to be equivalent to a classical Ising model in "$(d+1)$-dimensions". The extra dimension is continuous, or it involves a continuous limit. Many features of the Ising model apply, notably all correlation inequalities, which makes the quantum Ising model an attractive model to study.

The hamiltonian of the quantum Ising model is

$$H^{\text{qu.Is.}}_{\Lambda,\beta,h} = -\beta \sum_{\{x,y\} \in \mathcal{E}_\Lambda} S_x^{(3)} S_y^{(3)} - h \sum_{x \in \Lambda} S_x^{(1)}. \tag{7.1}$$

The equivalent $(d+1)$-dimensional model is defined on the set $\Lambda \times [0,1]_{\text{per}}$ with the extra dimension being the continuous interval $[0,1]$ with periodic boundary conditions. We often call this dimension "time" or "imaginary time". The reason is that such representations were first introduced by Dirac and Feynman for the evolution operator $\mathrm{e}^{-\mathrm{i}tH}$ in quantum mechanics; the case of the Gibbs operator $\mathrm{e}^{-\beta H}$ amounts to consider the imaginary time $t = -\mathrm{i}\beta$.

In this chapter we consider a discretised interval. For $n \in \mathbb{N}$, let

$$\Lambda_n = \Lambda \times \big\{0, \tfrac{1}{n}, \dots, \tfrac{n-1}{n}\big\}. \tag{7.2}$$

The set of edges consists of "horizontal" and "vertical" edges, $\mathcal{E}_{\Lambda_n} = \mathcal{E}^{\text{hor}}_{\Lambda_n} \cup \mathcal{E}^{\text{ver}}_{\Lambda_n}$, where

$$\begin{aligned} \mathcal{E}^{\text{hor}}_{\Lambda_n} &= \Big\{ \big\{ (x, \tfrac{i}{n}), (y, \tfrac{i}{n}) \big\} : \{x,y\} \in \mathcal{E}_\Lambda, i = 0,1,\dots,n-1 \Big\}, \\ \mathcal{E}^{\text{ver}}_{\Lambda_n} &= \Big\{ \big\{ (x, \tfrac{i}{n}), (x, \tfrac{i+1}{n}) \big\} : x \in \Lambda, i = 0,1,\dots,n-1 \Big\}. \end{aligned} \tag{7.3}$$

The vertical edges include those of the form $\{(x,0),(x,\frac{n-1}{n})\}$, that reflect the periodic boundary conditions of the extra dimension. This is illustrated in Fig. 7.1.

Next, we let $\boldsymbol{\omega} = (\omega^{(0)}, \omega^{(\frac{1}{n})}, \dots, \omega^{(\frac{n-1}{n})})$ denote "space-time configurations". Each $\boldsymbol{\omega}^{(\frac{i}{n})} \in \Omega_\Lambda$ is a usual Ising configuration, and we let $\Omega_{\Lambda_n} = \{-1,+1\}^{\Lambda_n}$ be the set of all space-time configurations $\boldsymbol{\omega}$. We now introduce the hamiltonian

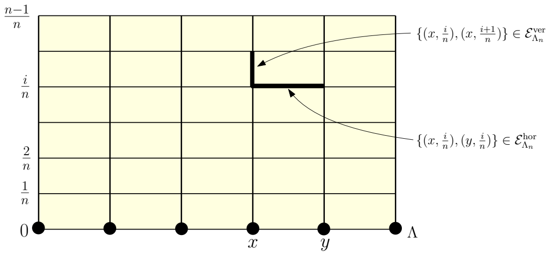



Figure 7.1. The $(d+1)$-dimensional "space-time" domain for the classical representation of the quantum Ising model. Two edges are shown in bold, an horizontal edge of $\mathcal{E}^{\rm hor}_{\Lambda_n}$ and a vertical edge of $\mathcal{E}^{\rm ver}_{\Lambda_n}$.

$H_{\Lambda_n} : \Omega_{\Lambda_n} \to \mathbb{R}$ defined as

$$H_{\Lambda_n}(\boldsymbol{\omega}) = -\tfrac{\beta}{4n} \sum_{\{(x,\frac{i}{n}),(y,\frac{i}{n})\}\in\mathcal{E}^{\rm hor}_{\Lambda_n}} \omega_x^{(\frac{i}{n})}\omega_y^{(\frac{i}{n})} - \big(-\tfrac12 \log\tfrac{h}{2n}\big) \sum_{\{(x,\frac{i}{n}),(x,\frac{i+1}{n})\}\in\mathcal{E}^{\rm ver}_{\Lambda_n}} \big(\omega_x^{(\frac{i}{n})}\omega_x^{(\frac{i+1}{n})} - 1\big). \tag{7.4}$$

This is a ferromagnetic Ising model indeed, with positive coupling parameters $\frac{\beta}{4n}$ for horizontal bonds, and $-\frac12 \log\frac{h}{2n}$ for vertical bonds. We Let $Z_{\Lambda_n}$ denote the classical partition function

$$Z_{\Lambda_n} = \sum_{\boldsymbol{\omega}\in\Omega_{\Lambda_n}} {\rm e}^{-H_{\Lambda_n}(\boldsymbol{\omega})}. \tag{7.5}$$

Let $f$ be a function $\Omega_\Lambda \to \mathbb{C}$ and $F$ the corresponding diagonal operator on $\mathcal{H}_\Lambda$, that is defined by

$$F|\omega\rangle = f(\omega)|\omega\rangle, \tag{7.6}$$

for all $\omega \in \Omega_\Lambda$. We can now state the precise relations between quantum Ising and the model above.

Theorem 7.1. *Assume that $\beta, h \geq 0$. Then*

(a) *Partition function:* ${\rm Tr}_{\mathcal{H}_\Lambda} {\rm e}^{-H^{\rm qu.Is.}_{\Lambda,\beta,h}} = \lim_{n\to\infty} Z_{\Lambda_n}$.

(b) *Gibbs state:* $\langle F\rangle^{\rm qu.Is.}_{\Lambda,\beta,h} = \lim_{n\to\infty} \frac{1}{Z_{\Lambda_n}} \sum_{\boldsymbol{\omega}\in\Omega_{\Lambda_n}} f(\omega^{(0)})\,{\rm e}^{-H_{\Lambda_n}(\boldsymbol{\omega})}$.

In item (b), $\omega^{(0)} = (\omega_x^{(0)})_{x\in\Lambda}$ is the configuration $\boldsymbol{\omega}$ restricted to the time 0. This theorem allows to derive properties of the quantum Ising model by looking at the classical representation. Compared to standard Ising, the main difficulty consists in checking that the results remain valid in the limit $n \to \infty$.



Proof of Theorem 7.1. We use Trotter's formula to get

$$\begin{aligned}{\rm Tr}_{\mathcal{H}_\Lambda} {\rm e}^{-H^{\rm qu.Is.}_{\Lambda,\beta,h}} &= \lim_{n\to\infty} {\rm Tr}_{\mathcal{H}_\Lambda} \Big( {\rm e}^{\frac{\beta}{n}\sum_{\|x,y\|\in\mathcal{E}_\Lambda} S_x^{(3)}S_y^{(3)}}\ {\rm e}^{\frac{h}{n}\sum_{x\in\Lambda} S_x^{(1)}} \Big)^n \\ &= \lim_{n\to\infty} \sum_{\boldsymbol{\omega}\in\Omega_{\Lambda_n}} \langle\omega^{(0)}|\,{\rm e}^{\frac{\beta}{n}\sum S_x^{(3)}S_y^{(3)}}\,|\omega^{(0)}\rangle\,\langle\omega^{(0)}|\,{\rm e}^{\frac{h}{n}\sum S_x^{(1)}}\,|\omega^{(\frac1n)}\rangle \\ &\qquad \dots \langle\omega^{(\frac{n-1}{n})}|\,{\rm e}^{\frac{\beta}{n}\sum S_x^{(3)}S_y^{(3)}}\,|\omega^{(\frac{n-1}{n})}\rangle\,\langle\omega^{(\frac{n-1}{n})}|\,{\rm e}^{\frac{h}{n}\sum S_x^{(1)}}\,|\omega^{(0)}\rangle. \end{aligned} \tag{7.7}$$

We used the standard definition for the trace, and standard matrix multiplication (except that indices are classical configurations); we also used the fact that terms involving the third spin operators are diagonal, namely

$$\langle\omega|\,{\rm e}^{\frac{\beta}{n}\sum S_x^{(3)}S_y^{(3)}}\,|\omega\rangle = {\rm e}^{\frac{\beta}{4n}\sum\omega_x\omega_y}. \tag{7.8}$$

The non-diagonal terms factorise for each site:

$$\langle\omega|\,{\rm e}^{\frac{h}{n}\sum_x S_x^{(1)}}\,|\omega'\rangle = \prod_{x\in\Lambda} \langle\omega_x|\,{\rm e}^{\frac{h}{n}S_x^{(1)}}\,|\omega'_x\rangle. \tag{7.9}$$

The inner products in the right side are in the space $\mathcal{H}_{\{x\}} \simeq \mathbb{C}^2$. We have

$$\begin{aligned}\langle\omega_x|\,{\rm e}^{\frac{h}{n}S_x^{(1)}}\,|\omega'_x\rangle &= \langle\omega_x|\big(1+\tfrac{h}{n}S_x^{(1)}\big)|\omega'_x\rangle + O(\tfrac{1}{n^2}) \\ &= \begin{cases} 1 + O(\frac{1}{n^2}) & \text{if } \omega_x = \omega'_x \\ \frac{h}{2n} + O(\frac{1}{n^2}) & \text{if } \omega_x \neq \omega'_x \end{cases} \\ &= {\rm e}^{-\frac12\log\frac{h}{2n}(\omega_x\omega'_x - 1)} + O(\tfrac{1}{n^2}). \end{aligned} \tag{7.10}$$

The last identity is easily verified by checking the two cases $\omega_x\omega'_x = \pm1$. Terms of order $\frac1n$ must be kept since they potentially occur $n$ times. Terms of order $\frac{1}{n^2}$ and lower can be discarded. We obtain

$$\begin{aligned}{\rm Tr}\,{\rm e}^{H^{\rm qu.Is.}_{\Lambda,\beta,h}} = \lim_{n\to\infty} \sum_{\boldsymbol{\omega}\in\Omega_{\Lambda_n}} \exp\Big\{ &\frac{\beta}{4n} \sum_{\{x,y\}\in\mathcal{E}_\Lambda} \sum_{i=0}^{n-1} \omega_x^{(\frac{i}{n})}\omega_y^{(\frac{i}{n})} \\ &+ \Big(-\tfrac12\log\tfrac{h}{2n}\Big) \sum_{x\in\Lambda}\sum_{i=0}^{n-1} \big(\omega_x^{(\frac{i}{n})}\omega_x^{(\frac{i+1}{n})}\big) \Big\}. \end{aligned} \tag{7.11}$$

This gives the expression (a) for the partition function.

The proof of (b) can be done in the same way, starting with an expression that is similar to Eq. (7.7), namely

$${\rm Tr}\,F\,{\rm e}^{-H^{\rm qu.Is.}_{\Lambda,\beta,h}} = \lim_{n\to\infty} \sum_{\boldsymbol{\omega}\in\Omega_{\Lambda_n}} f(\omega^{(0)})\langle\omega^{(0)}|\,{\rm e}^{\frac{\beta}{n}\sum S_x^{(3)}S_y^{(3)}}\,|\omega^{(0)}\rangle \dots \langle\omega^{(\frac{n-1}{n})}|\,{\rm e}^{\frac{h}{n}\sum S_x^{(1)}}\,|\omega^{(0)}\rangle. \tag{7.12}$$

The matrix elements can be replaced by ${\rm e}^{-H_{\Lambda_n}(\boldsymbol{\omega})}$ as before, which gives the expression (b). □

CHAPTER 8

# Graphical representations

The origin of graphical representations goes back to Feynman's description of the quantum Bose gas in terms of interacting Brownian trajectories [?]. In 1969, Ginibre proved the occurrence of phase transitions in quantum perturbations of classical models, using "space-time configurations" and a Peierls argument [**15**]. The quantum spin $\frac{1}{2}$ Heisenberg ferromagnet was described using random transpositions by Powers [?], and independently by Tóth [**33**]. Tóth used it to derive a bound for the free energy. Another loop representation was proposed by Aizenman and Nachtergaele for the quantum spin $\frac{1}{2}$ Heisenberg antiferromagnet [?]. This allowed them to relate the quantum chain to two classical two-dimensional models, namely the Potts and random cluster models. The synthesis of these representations, that applies to intermediate models such as quantum XY, was proposed more recently [**34**].

In this chapter, we restrict our attention to the case $S = \frac{1}{2}$, mainly for pedagogical reasons. The case $S = 1$ is very interesting, though.

We first describe the random loop models; the connection to quantum spins can be found in Section 2.

## 1. Random loop models

The Poisson point process plays an essential rôle. It can be introduced in many different ways including awfully abstract ones. We rather choose a pedestrian and intuitive approach.

The Poisson point process describes the occurrence of events that happen at any time and independently of each other. Let us discretise the time interval $[0, 1]$ by choosing $n \in \mathbb{N}$ and considering the set $\{0, \frac{1}{n}, \dots, \frac{n-1}{n}\}$. The random object is the subset $\eta \subset \{0, \frac{1}{n}, \dots, \frac{n-1}{n}\}$ that gives the set of occurrences. Let $\lambda > 0$ be the *intensity of the process*. The probability of the subset $\eta$ is defined to be

$$\mathbb{P}_n(\eta) = \big(\tfrac{\lambda}{n}\big)^{|\eta|} \big(1 - \tfrac{\lambda}{n}\big)^{n-|\eta|}. \tag{8.1}$$

We assume that $n$ is large enough so that $\frac{\lambda}{n} < 1$. The interpretation is that each point of the form $\frac{i}{n}$, $0 \leq i \leq n-1$, occurs with probability $\lambda/n$. As $n \to \infty$, this process converges to the Poisson point process on $[0, 1]$ with intensity $\lambda$.

The total number of events is not fixed; it is a Poisson random variable with parameter $\lambda$. Indeed, for $k \in \mathbb{N}_0$, we have

$$\mathbb{P}(|\eta| = k) = \tfrac{\lambda^k}{k!}\, \mathrm{e}^{-\lambda}\,. \tag{8.2}$$

Integration with respect to the Poisson point process can also be defined by a limit. Let $f = (f_k)$ be a collection of smooth functions $f_k : [0,1]^k \to \mathbb{R}$ (with $f_0 = 1$ by definition). Then

$$\begin{aligned} \mathbb{E}(f) &= \lim_{n\to\infty} \sum_{\eta} \big(\tfrac{\lambda}{n}\big)^{|\eta|} \big(1 - \tfrac{\lambda}{n}\big)^{n-|\eta|} f_{|\eta|}(\eta) \\ &= \lim_{n\to\infty} \sum_{k\geq 0} \big(\tfrac{\lambda}{n}\big)^{k} \big(1 - \tfrac{\lambda}{n}\big)^{n-k} \sum_{\eta, |\eta|=k} f_k(\eta) \\ &= \sum_{k\geq 0} \lambda^k \,\mathrm{e}^{-\lambda} \int_{0<t_1<\dots<t_k<1} f_k(t_1,\dots,t_k)\, \mathrm{d}t_1 \dots \mathrm{d}t_k. \end{aligned} \tag{8.3}$$

The exchange of limit and sum can be justified by dominated convergence. The latter expression is concrete and practical.

Next, we describe the model of random loops. Let $(\Lambda, \mathcal{E}_\Lambda)$ denote a finite graph, with $\Lambda$ the set of vertices and $\mathcal{E}_\Lambda$ the set of edges. Let $\beta > 0$ and $u \in [0,1]$ be two parameters. To each edge $\{x,y\} \in \mathcal{E}_\Lambda$ is associated the interval $[0,\beta]$ and a Poisson point process on this interval. There occur two kinds of events:

- "crosses" occur with intensity $u$;
- "double bars" occur with intensity $1-u$.

(This process can be defined by generalising the Poisson point process above to two kinds of events, or by considering two separate, independent processes.) We let $\rho$ denote the measure of independent Poisson point processes on $\mathcal{E}_\Lambda \times [0,\beta]$. Let $\eta$ denote realisations of this probability measure. It contains finitely many objects with probability 1.

To a given realisation $\eta$ corresponds a set of loops on $\mathcal{E}_\Lambda \times [0,\beta]$, denoted $\mathcal{L}(\eta)$. The loops consist of vertical lines connected by crosses or bars, with periodic boundary conditions in the continuous direction. This is best understood by looking at Fig. 8.1.

The *partition function* $Z^{\text{loops}}_{\beta,\Lambda}$ of the model of random loops is defined by

$$Z^{\text{loops}}_{\beta,\Lambda} = \int 2^{|\mathcal{L}(\eta)|} \rho(\mathrm{d}\eta). \tag{8.4}$$

The relevant measure for the model of random loops is given by

$$\mu(\mathrm{d}\eta) = \frac{1}{Z^{\text{loops}}_{\beta,\Lambda}} 2^{|\mathcal{L}(\eta)|} \rho(\mathrm{d}\eta). \tag{8.5}$$



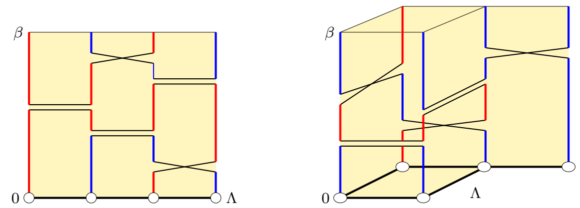


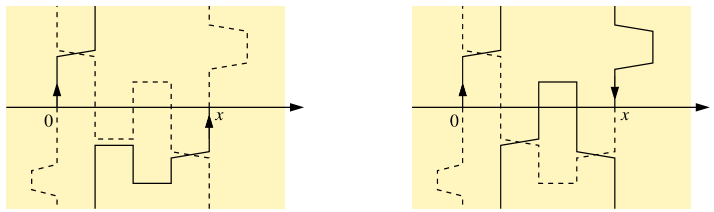

FIGURE 8.1. Graphs and realizations of Poisson point measures, and their loops. In both cases, the number of loops is $|\mathcal{L}(\eta)| = 2$.

It can be shown that for $\beta$ small the loops have small lengths and the probability that two sites belong to the same loop shows exponential decay with respect to the distance between the sites.

The loop correlations are given by the following three events:

- $E^+_{x,y}$ is the set of all realisations $\eta$ such that $x$ and $y$ belong to the same loop, and with identical vertical direction at these points.
- $E^-_{x,y}$ is the set of all $\eta$ such that $x$ and $y$ belong to the same loop, and with opposite vertical directions at these points.
- $E_{x,y} = E^+_{x,y} \cup E^-_{x,y}$ is the set of all $\eta$ such that $x$ and $y$ belong to the same loop.

These events are illustrated in Fig. 8.2. When $u = 1$, that is, when only crosses are present, we have $\mathbb{P}(E^+_{x,y}) = \mathbb{P}(E_{x,y})$. When $u = 0$, and if the graph is bipartite, we have $\mathbb{P}(E^+_{x,y}) = \mathbb{P}(E_{x,y})$ if $x,y$ belong to the same sublattice, and $\mathbb{P}(E^-_{x,y}) = \mathbb{P}(E_{x,y})$ if $x,y$ belong to different sublattices.




(a) (b)

FIGURE 8.2. Illustration for (a) the event $E^+_{0,x}$; (b) the event $E^-_{0,x}$. Here, we shifted the vertical direction and considered $\mathcal{E}_\Lambda \times [-\frac{\beta}{2}, \frac{\beta}{2}]$. Because of periodic boundary conditions in the vertical direction, this is equivalent to $\mathcal{E}_\Lambda \times [0,\beta]$.

## 2. Relations between random loops and quantum spins

We consider the following family of hamiltonians:

$$H_\Lambda^{(u)} = -2 \sum_{\{x,y\}\in\mathcal{E}_\Lambda} \Big( S_x^{(1)} S_y^{(1)} + (2u-1) S_x^{(2)} S_y^{(2)} + S_x^{(3)} S_y^{(3)} - \tfrac14 \Big). \tag{8.6}$$

With the parameter $u$ in the interval $[0,1]$, this family interpolates between the Heisenberg ferromagnet ($u=1$), the quantum XY model ($u=\frac12$), and the Heisenberg antiferromagnet (more precisely, it is unitarily equivalent to the case $u=0$ if the graph is bipartite). Notice that the inverse temperature $\beta$ does not appear in the hamiltonian. In this chapter we find it more convenient to put it in the Gibbs operator $\mathrm{e}^{-\beta H_\Lambda^{(u)}}$.

Here are precise relations between random loops and quantum spins.

THEOREM 8.1.

(i) *The partition functions of both models are identical:*

$$\mathrm{Tr}\, \mathrm{e}^{-\beta H_\Lambda^{(u)}} = \int 2^{|\mathcal{L}(\eta)|} \rho(\mathrm{d}\eta).$$

(ii) *Correlations in the spin directions 1 and 3 are given by loop correlations: For all $x,y\in\Lambda$,*

$$\langle S_x^{(1)} S_y^{(1)} \rangle_{\Lambda,\beta} = \langle S_x^{(3)} S_y^{(3)} \rangle_{\Lambda,\beta} = \tfrac14 \mathbb{P}(E_{x,y}).$$

(iii) *Correlations in the spin direction 2 are more subtle: For all $x,y\in\Lambda$,*

$$\langle S_x^{(2)} S_y^{(2)} \rangle_{\Lambda,\beta} = \tfrac14 \big[ \mathbb{P}(E_{x,y}^+) - \mathbb{P}(E_{x,y}^-) \big].$$

We prove this theorem by writing the Hamiltonian in terms of suitable interaction operators $T_{xy}$ and $Q_{xy}$, and by using a "Poisson expansion" of the Gibbs operator $\mathrm{e}^{-\beta H_\Lambda^{(u)}}$, see Lemma 8.3.

Let $T_{xy}$ be the transposition operator on $\mathcal{H}_{\{x,y\}} \simeq \mathbb{C}^4$:

$$T_{xy} |\omega_x, \omega_y\rangle = |\omega_y, \omega_x\rangle, \tag{8.7}$$

for all $\omega_x, \omega_y = \pm1$. And let $Q_{xy}$ be the operator on $\mathcal{H}_{\{x,y\}}$ with the following matrix elements:

$$\langle \omega_x, \omega_y | Q_{xy} | \omega_x', \omega_y' \rangle = \delta_{\omega_x,\omega_y} \delta_{\omega_x',\omega_y'}, \tag{8.8}$$

for all $\omega_x, \omega_y, \omega_x', \omega_y' = \pm1$. These operators can be expressed in terms of spin operators.

LEMMA 8.2. *Show that*

(i) $\vec S_x \cdot \vec S_y = \frac12 T_{xy} - \frac14$.

(ii) $S_x^{(1)} S_y^{(1)} - S_x^{(2)} S_y^{(2)} + S_x^{(3)} S_y^{(3)} = \frac12 Q_{xy} - \frac14$.



The proof of this lemma is done in Exercise 8.5. The Hamiltonian $H_\Lambda^{(u)}$ is a convex combination of these interactions, namely

$$H_\Lambda^{(u)} = - \sum_{\{x,y\}\in\mathcal{E}_\Lambda} \big( u T_{xy} + (1-u) Q_{xy} - 1 \big). \tag{8.9}$$

This is the perfect expression for a Feynman-Kac type of expansion.

LEMMA 8.3. *Let $A_1,\dots,A_k$ be bounded operators, and $\rho$ a Poisson point process on $\{1,\dots,k\}\times[0,1]$ of intensity 1. Then*

$$\exp\Big\{ \sum_{j=1}^k (A_j - 1) \Big\} = \int \rho(\mathrm{d}\eta) \prod_{(j,t)\in\eta}^{*} A_j.$$

In this lemma, the product $\prod^*$ is over the events of $\eta$ in increasing times.

PROOF. We start with the Poisson point process. We have

$$\begin{aligned}
\int \rho(\mathrm{d}\eta) \prod_{(j,t)\in\eta}^{*} A_j &= \lim_{n\to\infty} \sum_{m=0}^n \sum_{\substack{(j_1,t_1),\dots,(j_m,t_m)\\ t_1<t_2<\dots<t_m}} \big(\tfrac1n\big)^m \big(1-\tfrac1n\big)^{kn-m} \prod_{i=1}^m A_{j_i} \\
&= \lim_{n\to\infty} \prod_{t=1}^n \prod_{j=1}^k \big(1 - \tfrac1n + \tfrac1n A_j\big) \\
&= \lim_{n\to\infty} \Big( \prod_{j=1}^k \big[ 1 + \tfrac1n (A_j - 1) \big] \Big)^n \\
&= \lim_{n\to\infty} \Big( 1 + \tfrac1n \sum_{j=1}^k (A_j - 1) \Big)^n \\
&= \exp\Big\{ \sum_{j=1}^k (A_j - 1) \Big\}.
\end{aligned} \tag{8.10}$$

The last identity follows from the Trotter formula, see Proposition A.5. ☐

PROOF OF THEOREM 8.1. We start with the equivalence of the partition functions. Given a realisation $\eta$ of the Poisson point process on $\mathcal{E}_\Lambda\times[0,\beta]$, let $m=|\eta|$ denote the total number of crosses and double bars. From Lemma 8.3, we have

$$\begin{aligned}
\mathrm{e}^{-\beta H_\Lambda^{(u)}} &= \exp\Big\{ \beta \sum_{\{x,y\}\in\mathcal{E}_\Lambda} \big( u T_{xy} + (1-u) Q_{xy} - 1 \big) \Big\} \\
&= \int \rho(\mathrm{d}\eta) \prod_{\{x,y,t\}\in\eta}^{*} \left\{ \begin{matrix} T_{xy} \text{ if the event is a cross} \\ \text{or} \\ Q_{xy} \text{ if the event is a double bar} \end{matrix} \right\}.
\end{aligned} \tag{8.11}$$

We actually used a straightforward generalisation of the lemma with two Poisson processes of intensities $u$ and $1-u$, in the interval $[0,\beta]$ rather than $[0,1]$. Expanding in the basis of classical spin configurations, we get

$$\mathrm{Tr}\ \mathrm{e}^{-\beta H_\Lambda^{(u)}} = \int \rho(\mathrm{d}\eta) \sum_{\omega^{(1)},\dots,\omega^{(m)}\in\Omega_\Lambda} \langle\omega^{(1)}|R_{x_1y_1}|\omega^{(2)}\rangle\langle\omega^{(2)}|R_{x_2y_2}|\omega^{(3)}\rangle\dots\langle\omega^{(m)}|R_{x_my_m}|\omega^{(1)}\rangle, \tag{8.12}$$

where $R_{x_iy_i}$ is equal to either $T_{x_iy_i}$ or $Q_{x_iy_i}$, depending on $\eta$. Observe that the product of matrix elements is zero, unless $\omega_z^{(i)} = \omega_z^{(i+1)}$ for all $z \neq x_i, y_i$. Further, the spin values at $x_i, y_i$ satisfy

- $\omega_{x_i}^{(i+1)} = \omega_{y_i}^{(i)}$ and $\omega_{y_i}^{(i+1)} = \omega_{x_i}^{(i)}$ if $R_{x_iy_i} = T_{x_iy_i}$;
- $\omega_{x_i}^{(i)} = \omega_{y_i}^{(i)}$ and $\omega_{x_i}^{(i+1)} = \omega_{y_i}^{(i+1)}$ if $R_{x_iy_i} = Q_{x_iy_i}$.

Let us introduce "space-time spin configurations", which are piecewise constant functions $\boldsymbol{\omega} : [0,\beta] \to \Omega_\Lambda$. Given a realisation $\eta$ of the Poisson point process, let $\Sigma(\eta)$ be the set of space-time spin configurations such that $\boldsymbol{\omega}_{x,t}$ is constant along each loop of $\eta$. This is illustrated in Fig. 8.3.

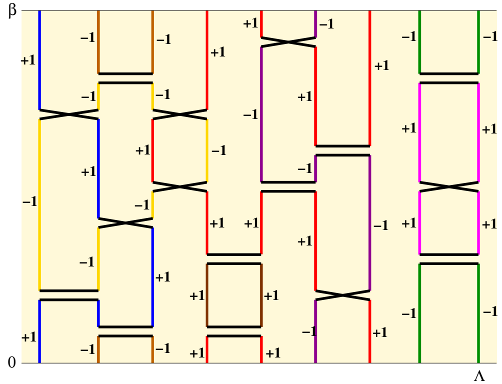



FIGURE 8.3. Illustration for a realisation of the measure $\rho$ and a compatible space-time spin configuration.

It is possible to rewrite Eq. (8.12) as

$$\mathrm{Tr}\ \mathrm{e}^{-\beta H_\Lambda^{(u)}} = \int \rho(\mathrm{d}\eta) \sum_{\boldsymbol{\omega}\in\Sigma(\eta)} 1. \tag{8.13}$$

There are two possibilities per loop, so the sum gives $2^{|\mathcal{L}(\eta)|}$. This proves (i).



For (ii), we expand as before and we get

$$\begin{aligned}\mathrm{Tr}\, S_x^{(3)} S_y^{(3)}\, \mathrm{e}^{-\beta H_\Lambda^{(u)}} &= \tfrac14 \int \rho(\mathrm{d}\eta) \sum_{\boldsymbol{\omega}\in\Sigma(\eta)} \boldsymbol{\omega}_{x,0}\boldsymbol{\omega}_{y,0} \\ &= \tfrac14 \int \rho(\mathrm{d}\eta) 1_{E_{xy}}(\eta) \underbrace{\sum_{\boldsymbol{\omega}\in\Sigma(\eta)} \boldsymbol{\omega}_{x,0}\boldsymbol{\omega}_{y,0}}_{=1} + \tfrac14 \int \rho(\mathrm{d}\eta) 1_{E_{xy}^{\mathrm{c}}}(\eta) \underbrace{\sum_{\boldsymbol{\omega}\in\Sigma(\eta)} \boldsymbol{\omega}_{x,0}\boldsymbol{\omega}_{y,0}}_{=0} \\ &= \tfrac14 \mathbb{P}(E_{xy}).\end{aligned} \tag{8.14}$$

The proof of (iii) is similar, but the operator $S_x^{(2)} S_y^{(2)}$ forces spin flips at $(x,0)$ and $(y,0)$. If $\eta$ does not contain a loop that connects $x$ and $y$, there are no compatible space-time spin configurations and we get 0. If $\eta \in E_{xy}^+$, the factor is

$$\langle \pm1 | S_x^{(2)} | \mp 1\rangle\, \langle \mp1 | S_y^{(2)} | \pm 1\rangle = \tfrac14. \tag{8.15}$$

If $\eta \in E_{xy}^-$, the factor is

$$\langle \pm1 | S_x^{(2)} | \mp 1\rangle\, \langle \pm1 | S_y^{(2)} | \mp 1\rangle = -\tfrac14. \tag{8.16}$$

It follows that $\langle S_x^{(2)} S_y^{(2)}\rangle$ involves the difference of probabilities of $E_{xy}^+$ and $E_{xy}^-$ and we obtain the identity (iii). □

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EXERCISE 8.1.

(i) *Check that the moment generating function (Laplace transform) of a Poisson random variable $X$ is given by*

$$\mathbb{E}(\,\mathrm{e}^{tX}\,) = \mathrm{e}^{\lambda(\,\mathrm{e}^t-1)}.$$

(b) *Let $\eta$ denote the discrete Poisson process defined in Eq. (8.1). Find the moment generating function of $|\eta|$, and check that*

$$\lim_{n\to\infty} \mathbb{E}_n(\,\mathrm{e}^{t|\eta|}\,) = \mathrm{e}^{\lambda(\,\mathrm{e}^t-1)}.$$

(iii) *Find $\mathbb{E}(X)$ and $\mathbb{E}_n(|\eta|)$. (Moment generating functions may help.)*

EXERCISE 8.2. *Let $\eta$ denote realisations of crosses and double bars on $\mathcal{E}_\Lambda \times [0,\beta]$. Check that $|\eta|$ is a Poisson random variable with parameter $\beta|\mathcal{E}_\Lambda|$.*

EXERCISE 8.3. *Let $H_\Lambda^{\mathrm{Heis.AF}} = 2\sum_{\{x,y\}\in\mathcal{E}_\Lambda} \vec{S}_x \cdot \vec{S}_y$. Find the unitary operator $U$ such that*

$$U H_\Lambda^{\mathrm{Heis.AF}} U^{-1} = H_\Lambda^{(u=0)}.$$

EXERCISE 8.4.

(i) *Check that $T_{x,y}$ is hermitian.*

(ii) *Find the eigenvalues of $T_{x,y}$ and their multiplicities. (Hint: Look at $T_{x,y}^2$ and $\operatorname{Tr} T_{x,y}$.)*

EXERCISE 8.5. *Prove Lemma 8.2.*

APPENDIX A

# Matrix inequalities

We collect here a series of useful properties of square matrices. Recall that the "absolute value" of a matrix is $|A| = (A^*A)^{\frac{1}{2}}$, where the square root of a nonnegative hermitian matrix can be defined by diagonalising and taking the square root of the eigenvalues. The $p$-norm of a matrix is then defined as

$$\|A\|_p = (\operatorname{Tr} |A|^p)^{1/p}. \tag{A.1}$$

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EXERCISE A.1.

(i) *Show that $\|A\|_p$ is decreasing in $p$, and that $\lim_{p\to\infty} \|A\|_p = \|A\|$.*
(ii) *Prove that $\|A\|_p$ is a norm for all $1 \leq p \leq \infty$.*
(iii) *Use Hölder inequality to show that $\|A\|_p$ is submultiplicative, that is, $\|AB\|_p \leq \|A\|_p \|B\|_p$.*

---

PROPOSITION A.1 (Hölder inequality for matrices). *If $1 \leq p, q, r \leq \infty$ with $\frac{1}{p} + \frac{1}{q} = \frac{1}{r}$, we have*

$$\|AB\|_r \leq \|A\|_p \|B\|_q.$$

It follows from a simple induction that

$$\Big\| \prod_{j=1}^n A_j \Big\|_r \leq \prod_{j=1}^n \|A_j\|_{p_j} \tag{A.2}$$

whenever $1 \leq r, p_1, \dots, p_n$ with $\sum_{j=1}^n \frac{1}{p_j} = \frac{1}{r}$. There are no short proofs of Hölder's inequality for matrices. The proof is due to Fröhlich [1978] and it uses chessboard estimates. The proof of Proposition A.1 can be found after Lemma that of A.4.

LEMMA A.2 (Chessboard estimate). *For any $n \in \mathbb{N}$ and any matrices $A_1, \dots, A_{2n}$, we have*

$$\big|\operatorname{Tr} A_1 \dots A_{2n}\big| \leq \prod_{i=1}^{2n} \Big(\operatorname{Tr} (A_i A_i^*)^n\Big)^{1/2n}.$$

PROOF. Since $(A,B) \mapsto \operatorname{Tr} A^* B$ is an inner product, the following inequality follows from Cauchy-Schwarz:

$$\big|\operatorname{Tr} A_1 \dots A_{2n}\big|^2 \leq \operatorname{Tr}\big(A_1 \dots A_n A_n^* \dots A_1^*\big) \operatorname{Tr}\big(A_{2n}^* \dots A_{n+1}^* A_{n+1} \dots A_{2n}\big). \tag{A.3}$$

This allows to use a reflection positivity argument. It is enough to prove the inequality for matrices that satisfy $\operatorname{Tr}(A_i A_i^*)^n = 1$; the general result follows from scaling.

Let $A_1, \dots, A_{2n}$ be matrices that maximise $|\operatorname{Tr} A_1 \dots A_{2n}|$, with maximum number of matching neighbours $A_{i+1} = A_i^*$. Suppose there exists an index $j$ such that $A_{j+1} \neq A_j^*$. Using cyclicity, we can assume that $j = n$. By the inequality (A.3), $A_1, \dots, A_n, A_n^*, \dots, A_1^*$ and $A_{2n}^*, \dots, A_{n+1}^*, A_{n+1}, \dots, A_{2n}$ are also maximisers. At least one has strictly more matching neighbours, hence a contradiction. The maximum is then $\operatorname{Tr}(AA^*)^n$ for some matrix $A$, which is equal to 1. □

Chessboard estimates allow to prove what is essentially the case $r = 1$ of Hölder's inequality.

COROLLARY A.3. *We have*

$$|\operatorname{Tr} A_1 \dots A_n| \leq \prod_{i=1}^n \|A_i\|_{p_i}$$

*for all $n$ and all rational $p_i$'s such that $\sum_{i=1}^n \frac{1}{p_i} = 1$.*

PROOF. Let $\ell$ be a positive integer such that $2\ell/p_i$ is integer for all $i$. Let $A_i = U_i|A_i|$ be the polar decomposition of $A_i$, and let

$$B_i = |A_i|^{p_i/2\ell}, \qquad \hat{B}_i = U_i |A_i|^{p_i/2\ell}. \tag{A.4}$$

Then $A_i = \hat{B}_i B_i^{(2\ell/p_i)-1}$, and we have

$$\begin{aligned} \big|\operatorname{Tr} A_1 \dots A_n\big| &= \big|\operatorname{Tr} \hat{B}_1 \underbrace{B_1 \dots B_1}_{(2\ell/p_1)-1} \dots \hat{B}_n \underbrace{B_n \dots B_n}_{(2\ell/p_n)-1}\big| \\ &\leq \prod_{i=1}^n (\operatorname{Tr} |A_i|^{p_i})^{1/p_i} \qquad \text{(A.5)} \\ &= \prod_{i=1}^n \|A_i\|_{p_i}. \end{aligned}$$

The inequality follows from Lemma A.2 and from the identities

$$\operatorname{Tr}(B_i B_i^*)^\ell = \operatorname{Tr}(\hat{B}_i \hat{B}_i^*)^\ell = \operatorname{Tr} |A_i|^{p_i}. \tag{A.6}$$

□



LEMMA A.4. *Let $r, r' \in [1,\infty]$ such that $\frac{1}{r} + \frac{1}{r'} = 1$. Then for any square matrix $A$, we have*

$$\|A\|_r = \sup_{\|C\|_{r'}=1} \operatorname{Tr} C^* A.$$

PROOF. The right side is smaller by Corollary A.3:

$$\big|\operatorname{Tr} C^* A\big| \leq \|C\|_{r'} \|A\|_r = \|A\|_r. \tag{A.7}$$

In order to check that this inequality is saturated, let $A = U|A|$ be the polar decomposition of $A$, and choose $C = \|A\|_r^{1-r} U |A|^{r-1}$. Then $\|C\|_{r'} = 1$ and $\operatorname{Tr} C^* A = \|A\|_r$. □

PROOF OF PROPOSITION A.1. Starting with Lemma A.4 and then using Corollary A.3, we have

$$\begin{aligned} \|AB\|_r &= \sup_{\|C\|_{r'}=1} \operatorname{Tr} C^* AB \\ &\leq \sup_{\|C\|_{r'}=1} \|C\|_{r'} \|A\|_p \|B\|_q. \end{aligned} \tag{A.8}$$

□

PROPOSITION A.5 (Trotter formula). *Let $A, B$ be square matrices. Then*

$$\mathrm{e}^{A+B} = \lim_{n\to\infty} \Big( \mathrm{e}^{\frac{1}{n}A}\, \mathrm{e}^{\frac{1}{n}B} \Big)^n = \lim_{n\to\infty} \Big[ \big(1 + \tfrac{1}{n}A\big) \mathrm{e}^{\frac{1}{n}B} \Big]^n.$$

PROOF. We prove the second formula — the mild changes for the first formula are straightforward. Let $K_n$ be the matrix such that

$$\big(1 + \tfrac{1}{n}A\big) \mathrm{e}^{\frac{1}{n}B} = 1 + \tfrac{1}{n}(A+B) + K_n. \tag{A.9}$$

It is clear that $\|K_n\| = O(\frac{1}{n^2})$. We have

$$\Big[\big(1 + \tfrac{1}{n}A\big) \mathrm{e}^{\frac{1}{n}B}\Big]^n = \Big(1 + \tfrac{1}{n}(A+B)\Big)^n + R_n, \tag{A.10}$$

where $R_n$ is a matrix whose norm satisfies

$$\|R_n\| \leq \sum_{k=0}^{n-1} \tbinom{n}{k} \|1 + \tfrac{1}{n}(A+B)\|^k \, \|K_n\|^{n-k} = O(\tfrac{1}{n}). \tag{A.11}$$

The first term in the right side of (A.10) converges to $\mathrm{e}^{A+B}$. □

The Duhamel formula allows to expand $e^{A+B}$ as powers of $B$.

PROPOSITION A.6 (Duhamel formula). *Let $A, B$ be $n \times n$ matrices. Then*

$$\begin{aligned} e^{A+B} &= e^{A} + \int_0^1 e^{tA} B\, e^{(1-t)(A+B)}\, dt \\ &= \sum_{k\geq 0} \int_{0<t_1<\cdots<t_k<1} dt_1 \dots dt_k\, e^{t_1 A} B\, e^{(t_2-t_1)A} B \dots B\, e^{(1-t_k)A}. \end{aligned}$$

PROOF. Let $F(s)$ be the matrix-valued function

$$F(s) = e^{sA} + \int_0^s e^{tA} B\, e^{(s-t)(A+B)}\, dt. \tag{A.12}$$

We show that, for all $s$,

$$e^{s(A+B)} = F(s). \tag{A.13}$$

The derivative of $F(s)$ is

$$F'(s) = e^{sA} A + e^{sA} B + \int_0^s e^{tA} B\, e^{(s-t)(A+B)} (A+B) dt = F(s)(A+B). \tag{A.14}$$

On the other hand, the derivative of $e^{s(A+B)}$ is $e^{s(A+B)}(A+B)$. The identity (A.13) clearly holds for $s = 0$ and, since both sides satisfy the same differential equation, they must be equal for all $s$.

We can iterate Duhamel's formula $N$ times so as to get

$$\begin{aligned} e^{A+B} &= \sum_{k=0}^{N} \int_{0<t_1<\cdots<t_k<1} dt_1 \dots dt_k\, e^{t_1 A} B\, e^{(t_2-t_1)A} B \dots B\, e^{(1-t_k)A} \\ &+ \int_{0<t_1<\cdots<t_N<1} dt_1 \dots dt_k\, e^{t_1 A} B\, e^{(t_2-t_1)A} B \dots B \Big[ e^{(1-t_N)(A+B)} - e^{(1-t_N)A} \Big]. \end{aligned} \tag{A.15}$$

Using $\| e^{tA} \| \leq e^{t\|A\|}$, the last line is less than $2\, e^{\|A\|+\|B\|} \frac{\|B\|^N}{N!}$ and so it vanishes in the limit $N \to \infty$. The summand is less than $e^{\|A\|} \frac{\|B\|^k}{k!}$, so that the sum is absolutely convergent. □

PROPOSITION A.7 (Golden-Thompson inequality). *Let $A, B$ be hermitian matrices. Then*

$$\mathrm{Tr}\, \big( e^{A+B} \big) \leq \mathrm{Tr}\, \big( e^{A}\, e^{B} \big).$$



PROOF. Hölder's inequality, Proposition A.1, implies that $\mathrm{Tr}\,(AB)^n \leq \|AB\|_n^n$. The latter is equal to $\mathrm{Tr}\,(A^2B^2)^{n/2}$ since $A, B$ are hermitian. If $n$ is a power of 2, we can iterate and we get

$$\mathrm{Tr}\,(AB)^n \leq \mathrm{Tr}\, A^n B^n. \tag{A.16}$$

We use this inequality with $A \mapsto e^{\frac{1}{n}A}$ and $B \mapsto e^{\frac{1}{n}B}$, which gives

$$\mathrm{Tr}\, \Big( e^{\frac{1}{n}A}\, e^{\frac{1}{n}B} \Big)^n \leq \mathrm{Tr}\, e^{A}\, e^{B}. \tag{A.17}$$

The left side converges to $\mathrm{Tr}\, e^{A+B}$ as $n \to \infty$ by the Trotter formula. □

PROPOSITION A.8 (Klein inequality). *Let $f$ be a convex differentiable function, and $A, B$ be hermitian matrices with eigenvalues in the domain of $f$. Then*

$$\mathrm{Tr}\, \big[ f(A) - f(B) - (A-B) f'(B) \big] \geq 0.$$

With $f(s) = e^s$, exchanging $A$ and $B$, we get

$$\mathrm{Tr}\, \big( e^{A} - e^{B} \big) \leq \mathrm{Tr}\, (A-B)\, e^{A}. \tag{A.18}$$

PROOF. Let $(\phi_i)$ and $(\psi_i)$ be orthonormal bases of eigenvectors of $A$ and $B$, and let $(a_i)$ and $(b_i)$ the eigenvalues. Let $c_{ij} = (\phi_i, \psi_j)$. Then

$$\begin{aligned} \big(\phi_i, \big[f(A) - f(B) - (A-B)f'(B)\big]\phi_i\big) &= f(a_i) - \sum_j |c_{ij}|^2 f(b_j) - \sum_j |c_{ij}|^2 (a_i - b_j) f'(b_j) \\ &= \sum_j |c_{ij}|^2 \big[ f(a_i) - f(b_j) - (a_i - b_j) f'(b_j) \big] \\ &\geq 0. \end{aligned} \tag{A.19}$$

□

APPENDIX B

# Solutions to some exercises

Exercise 2.7: Since $f(\mathbb{1}) = 1$, it is clear that $\|f\| \geq 1$. We show that $|\omega(a)| \leq 1$ for all $a$ in the Hilbert space with $\|a\| = 1$. If $a = a^*$, we have $1 \pm a \geq 0$ and $f(1 \pm a) = 1 \pm f(a) \geq 0$, so that $|f(a)| \leq 1$.

For general $a$, we have

$$f\big((a^* - \overline{f(a)})(a - f(a))\big) \geq 0 \tag{B.1}$$

and linearity implies that $|f(a)|^2 \leq f(a^*a) \leq 1$.

Exercise 2.8: Given two operators $a, b$, let $\langle a, b\rangle = \operatorname{Tr} a^* b$. This map is an inner product, which turns the vector space of operators into a Hilbert space. The standard Riesz representation theorem implies that this space is self-dual, that is, every linear functional $f$ is represented by a unique operator $\rho$ so that

$$f(a) = \langle \rho, a\rangle = \operatorname{Tr} \rho^* a.$$

There remains to check that $\rho$ is a density operator. We have for all vector $\varphi$ with $\|\varphi\| = 1$ that

$$\langle \varphi, \rho^* \varphi\rangle = \operatorname{Tr} P_\varphi \rho^* = f(P_\varphi) = f(P_\varphi^2) \geq 0.$$

Then $\rho^*$ is positive-definite; it is therefore hermitian so $\rho \geq 0$ as well. Finally, $1 = f(\mathbb{1}) = \operatorname{Tr} \rho \mathbb{1}$, so $\rho$ is indeed a density operator.

Exercise 2.9: Let $\rho$ be a minimiser with $\operatorname{Tr} \rho = 1$. Let $\alpha \in [-1, 1]$ and $\eta$ be any operator such that $\operatorname{Tr} \eta = 0$ and $\rho \pm \eta \geq 0$. Since $F(\rho + \alpha\eta)$ is minimal at $\alpha = 0$, we get

$$0 = \frac{\mathrm{d}}{\mathrm{d}\alpha} \Big[ \operatorname{Tr} H(\rho + \alpha\eta) + \frac{1}{\beta} \operatorname{Tr} (\rho + \alpha\eta) \log(\rho + \alpha\eta) \Big] \Big|_{\alpha=0} = \operatorname{Tr} \eta \Big( H + \frac{1}{\beta} \log \rho \Big).$$

Then $H + \frac{1}{\beta} \log \rho$ is proportional to the identity operator, so $\rho$ is proportional to $\mathrm{e}^{-\beta H}$.

Notice that $\frac{\mathrm{d}^2}{\mathrm{d}\alpha^2} F(\rho + \alpha\eta) = \frac{1}{\beta} \operatorname{Tr} \eta \rho^{-1} \eta \geq 0$, so $\alpha = 0$ is a minimum indeed.

Exercise 2.3: Part (i) follows from linearity and the commutation relations for $S^1, S^2, S^3$.

For part (ii), we replace $\vec{a}$ by $s\vec{a}$, and we check that both sides of the identity satisfy the same differential equation. We find

$$\frac{\mathrm{d}}{\mathrm{d}s} \, \mathrm{e}^{-\mathrm{i} S^{s\vec{a}}} \, S^{\vec{b}} \, \mathrm{e}^{\mathrm{i} S^{s\vec{a}}} = -\mathrm{i} [S^{\vec{a}}, \, \mathrm{e}^{-\mathrm{i} S^{s\vec{a}}} \, S^{\vec{b}} \, \mathrm{e}^{\mathrm{i} S^{s\vec{a}}}], \tag{B.2}$$

and

$$\frac{\mathrm{d}}{\mathrm{d}s} S^{R_{s\vec{a}}\vec{b}} = \Big(\frac{\mathrm{d}}{\mathrm{d}s} R_{s\vec{a}}\vec{b}\Big) \cdot \vec{S} = \Big(\vec{a} \times R_{s\vec{a}}\vec{b}\Big) \cdot \vec{S} = -\mathrm{i}[S^{\vec{a}}, S^{R_{s\vec{a}}\vec{b}}]. \tag{B.3}$$

We used part (i) for the last identity.

Exercise 2.4: Let us define the unitary operator

$$U_\Lambda = \prod_{x\in\Lambda} \mathrm{e}^{\mathrm{i}\pi S_x^{(1)}}.$$

All hamiltonians (with $h = 0$) are invariant under this transformation, i.e. $U_\Lambda^{-1} H_{\Lambda,\beta,0} U_\Lambda = H_{\Lambda,\beta,0}$, so that

$$\langle S_x^{(3)} \rangle_{\Lambda,\beta,0} = \langle U_\Lambda S_x^{(3)} U_\Lambda^{-1} \rangle_{\Lambda,\beta,0} = -\langle S_x^{(3)} \rangle_{\Lambda,\beta,0}.$$

Then $\langle S_x^{(3)} \rangle_{\Lambda,\beta,0} = 0$. For the other identities, consider $U_\Lambda = \prod_{x\in\Lambda} \mathrm{e}^{\mathrm{i}\pi S_x^{(3)}}$. Then $U_\Lambda^{-1} H_{\Lambda,\beta,h} U_\Lambda = H_{\Lambda,\beta,h}$ and $\langle S_x^{(i)} \rangle_{\Lambda,\beta,h} = -\langle S_x^{(i)} \rangle_{\Lambda,\beta,h}$ for $i = 1, 2$.

Exercise 5.1: Let $C_0$ a connected set of size $|C_0| = n \geq 2$ that contains the origin. There exists a walk $(0, x_1, x_2, \dots, x_{2(n-1)})$ with $\|x_i - x_{i+1}\|_1 = 1$ whose support is precisely $C_0$. Indeed, this is clearly true for $n = 2$, and this can be extended to a larger set by induction: Knowing the walk for $C_0$, we get a walk for $C_0 \cup \{x\}$ by making an excursion to $x$ and back; this increases the length by 2. Now we consider all walks that start at the origin, of length $2(n-1)$, and look at the support. All sets of size $n$ will appear at least once (most of them will appear many times, in fact).

# Bibliography


[1] I. Affleck, T. Kennedy, E.H. Lieb, H. Tasaki, *Valence bond ground states in isotropic quantum antiferromagnets*, Comm. Math. Phys. 115, 477-528 (1988)

[2] M. Aizenman, B. Nachtergaele, *Geometric aspects of quantum spin states*, Comm. Math. Phys. 164, 17–63 (1994)

[3] M. Biskup, *Reflection positivity and phase transitions in lattice spin models*, in Methods of contemporary mathematical statistical physics, Lect. Notes Math. 1970, 1–86 (2009)

[4] J. Björnberg, *Infrared bounds and mean-field behaviour in the quantum Ising model*, Commun. Math. Phys. 323, 329–366 (2013)

[5] J. Björnberg, *Vanishing critical magnetization in the quantum Ising model*, Commun. Math. Phys. 337, 879–907 (2015)

[6] J. Björnberg, *Fermionic observables in the transverse Ising chain*, J. Math. Phys. 58, 053302 (2017)

[7] J. Björnberg, G.R. Grimmett, *The phase transition of the quantum Ising model is sharp*, J. Stat. Phys. 136, 231 (2009)

[8] J. Conlon, J.P. Solovej, *Upper bound on the free energy of the spin 1/2 Heisenberg ferromagnet*, Lett. Math. Phys. 23, 223–231 (1991)

[9] N. Crawford, D. Ioffe, *Random current representation for transverse field Ising model*, Comm. Math. Phys. 296, 447–474 (2010)

[10] F.J. Dyson, E.H. Lieb, B. Simon, *Phase transitions in quantum spin systems with isotropic and nonisotropic interactions*, J. Statist. Phys. 18, 335–383 (1978)

[11] M.E. Fisher, D. Jasnow, *Decay of order in isotropic systems of restricted dimensionality. II. Spin systems*, Phys. Rev. B 3, 907–924 (1971)

[12] J. Fröhlich, *Mathematical physics of phase transitions and symmetry breaking*, Bull. Amer. Math. Soc. 84, 165–193 (1978)

[13] J. Fröhlich, B. Simon, T. Spencer, *Infrared bounds, phase transitions and continuous symmetry breaking*, Comm. Math. Phys. 50, 79–95 (1976)

[14] S. Friedli and Y. Velenik. *Statistical Mechanics of Lattice Systems: a Concrete Mathematical Introduction*, Cambridge University Press (2017)

[15] J. Ginibre, *Existence of phase transitions for quantum lattice systems*, Comm. Math. Phys. 14, 205– (1969)

[16] C. Goldschmidt, D. Ueltschi, P. Windridge, *Quantum Heisenberg models and their probabilistic representations*, in Entropy and the Quantum II, Contemp. Math. 552, 177–224 (2011); arXiv:1104.0983 [math-ph]

[17] G.R. Grimmett, *Space-time percolation*, in In and out of equilibrium 2, Progr. Probab. 60, 305–320 (2008)

[18] D. Ioffe, *Stochastic geometry of classical and quantum Ising models*, in Methods of contemporary mathematical statistical physics, Lect. Notes Math. 1970, 87–127 (2009)

[19] R.B. Israel, *Convexity in the Theory of Lattice Gases*, Princeton University Press (1979)

[20] T. Kennedy, *Long range order in the anisotropic quantum ferromagnetic Heisenberg model*, Comm. Math. Phys. 100, 447–462 (1985)
[21] T. Kennedy, E.H. Lieb, B.S. Shastry, *Existence of Néel order in some spin-$\frac{1}{2}$ Heisenberg antiferromagnets*, J. Statist. Phys. 53, 1019–1030 (1988)
[22] T. Kennedy, E.H. Lieb, B.S. Shastry, *The $XY$ model has long-range order for all spins and all dimensions greater than one*, Phys. Rev. Lett. 61, 2582–2584 (1988)
[23] T. Koma, H. Tasaki, *Decay of superconducting and magnetic correlations in one-and two-dimensional Hubbard models*, Phys. Rev. Lett. 68, 3248–3251 (1992)
[24] J.H. Li, *The interface of the FK-representation of the quantum Ising model converges to the SLE16/3*, arXiv:1608.02821 (2016)
[25] E. H. Lieb, *The classical limit of quantum spin systems*, Comm. Math. Phys. 31, 327–340 (1973)
[26] O.A. McBryan, T. Spencer, *On the decay of correlations in $SO(n)$-symmetric ferromagnets*, Comm. Math. Phys. 53, 299–302 (1977)
[27] N.D. Mermin, H. Wagner, *Absence of ferromagnetism or antiferromagnetism in one- or two-dimensional isotropic Heisenberg models*, Phys. Rev. Lett. 17, 1133–1136 (1966)
[28] B. Nachtergaele, *Quasi-state decompositions for quantum spin systems*, in Probability Theory and Mathematical Statistics, B Grigelionis et al (eds), pp 565–590 (1994); arXiv:cond-mat/9312012
[29] B. Nachtergaele, *A stochastic geometric approach to quantum spin systems*, in Probability and Phase Transitions, G. Grimmett (ed.), Nato Science series C 420, pp 237–246 (1994)
[30] E.J. Neves, J.F. Perez, *Long range order in the ground state of two-dimensional antiferromagnets*, Phys. Lett. A 114, 331–333 (1986)
[31] P. Pfeuty, *The one-dimensional Ising model with a transverse field*, Ann. Phys. 57(1), 79–90 (1970)
[32] B. Simon, *The Statistical Mechanics of Lattice Gases*, Princeton University Press (1993)
[33] B. Tóth, *Improved lower bound on the thermodynamic pressure of the spin* 1/2 *Heisenberg ferromagnet*, Lett. Math. Phys. 28, 75–84 (1993)
[34] D. Ueltschi, *Random loop representations for quantum spin systems*, J. Math. Phys. 54, 083301, 1–40 (2013)

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