Notes for the support class of the course MA4L2

March 20, 2017

**Exercise 2.1.** Let $P_\epsilon$ be the projector onto the subspace of $H_\Lambda$ spanned by configurations $\omega$ such that $\frac{1}{|\Lambda|} \sum_{x \in \Lambda} \omega_x \notin (-\epsilon, \epsilon)$. Suppose that $[H_{\Lambda, \beta, 0}, \sum_{x \in \Lambda} S_3^x] = 0$. Show that for any sequence $\Lambda_n \uparrow \mathbb{Z}^d$, if $\psi(\beta, h)$ is differentiable in $h$ at $(\beta, 0)$, then for $\epsilon > 0$, and for large enough $n$:

\[
\langle P_\epsilon \rangle_{\Lambda_n, \beta, 0} \leq e^{-c|\Lambda_n|}
\]

for some $c > 0$ uniform in $\Lambda_n$. (Hint: Show that a Chernov inequality holds.)

**Proof.** Firstly, we can rewrite the projector in the following way:

\[
P_\epsilon = P_\epsilon^+ + P_\epsilon^-
\]

where the two projectors on the right hand side project over the subspaces of $H_{\Lambda_n}$ spanned by configurations $\omega$ such that $\frac{1}{|\Lambda_n|} \sum_{x \in \Lambda_n} \omega_x \geq \epsilon$ (or respectively $\leq -\epsilon$). Let $\mathcal{H}^\epsilon_{\Lambda_n}$ (respectively $\mathcal{H}^-_{\Lambda_n}$) denote such spaces. We are going to show only that the result holds for $\langle P_\epsilon^+ \rangle_{\Lambda_n, \beta, 0}$ - the result for the projector on $\mathcal{H}^-_{\Lambda_n}$ follows in a similar way. Notice that for any $h > 0$, $P_\epsilon^+ \leq e^{-\frac{\epsilon}{2} |\Lambda_n|} e^{h \sum_x S_x^3}$. Indeed, we have:

\[
\langle \omega | e^{-\frac{\epsilon}{2} |\Lambda_n|} e^{h \sum_x S_x^3} | \omega \rangle = e^{-\frac{\epsilon}{2} |\Lambda_n|} h^\frac{b}{2} \sum_x \omega_x \left\{ \begin{array}{ll} 
\geq 1 & \text{if } \omega \in \mathcal{H}^\epsilon_{\Lambda_n} \\
 \in (0, 1) & \text{otherwise.} \end{array} \right.
\]

Then it follows that

\[
\langle P_\epsilon^+ \rangle_{\Lambda_n, \beta, 0} \leq \langle e^{-\frac{\epsilon}{2} |\Lambda_n|} e^{h \sum_x S_x^3} \rangle_{\Lambda_n, \beta, 0} = e^{-\frac{\epsilon}{2} |\Lambda_n|} Z_{\Lambda_n, \beta, h} / Z_{\Lambda_n, \beta, 0} = e^{-\frac{\epsilon}{2} |\Lambda_n|} e^{\psi(\beta, h) - \psi(\beta, 0)}
\]

(4)
In order to prove the result, it is sufficient to show that:

$$
\limsup_{n \to \infty} \frac{1}{|\Lambda_n|} \log \langle P_{+\epsilon} \rangle_{\Lambda_n,\beta,0} \leq -C,
$$

(5)

where $C$ is some positive constant. Notice that:

$$
\limsup_{n \to \infty} \frac{1}{|\Lambda_n|} \log \langle P_{+\epsilon} \rangle \leq -\lim_{n \to \infty} \left( \frac{\epsilon h}{2} - (\psi_{\Lambda_n}(\beta, h) - \psi_{\Lambda_n}(\beta, 0)) \right)
$$

(6)

By a Taylor expansion we have:

$$
\frac{\epsilon}{2} - (\psi(\beta, h) - \psi(\beta, 0)) = h \frac{\epsilon}{2} \left( 1 - 2 \frac{\partial}{\partial h} \psi(\beta, \hat{h}) \right),
$$

(7)

with $\hat{h} \in (0, h)$. Remember that $h$ is arbitrary – so we can then fix a value of $h$ such that the right hand side of the equation above is positive. Such a value exists because $\frac{\partial}{\partial h} \psi(\beta, 0) = 0$ and $\psi(\beta, h)$ is convex.

Exercise 3.1. Let $h > 0$. Show that $\langle \prod_{x \in X} S_x^3 \rangle_{\Lambda,\beta,h} \geq 0$ for all $X \subset \Lambda$, where $\langle \cdot \rangle_{\Lambda,\beta,h}$ is the Gibbs state related to the Heisenberg hamiltonian.

Proof. Recall the explicit formulation of the Heisenberg hamiltonian:

$$
H^{\text{heis}}_{\Lambda,\beta,h} = -\beta \sum_{\{x,y\} \in E_{\Lambda}} S_x^1 S_y^1 + S_x^2 S_y^2 + S_x^3 S_y^3 - h \sum_{x \in \Lambda} S_x^3.
$$

(8)

Let us define the following operator:

$$
U = \prod_{x \in \Lambda} e^{i\frac{\pi}{2} S_x^2}.
$$

(9)

Notice that by the result in Exercise 1.3 of the lecture notes

$$
e^{-i\frac{\pi}{2} S_x^2} S_x^1 e^{i\frac{\pi}{2} S_x^2} = -S_x^3,
$$

(10)

$$
e^{-i\frac{\pi}{2} S_x^2} S_x^2 e^{i\frac{\pi}{2} S_x^2} = S_x^2,
$$

(11)

$$
e^{-i\frac{\pi}{2} S_x^2} S_x^3 e^{i\frac{\pi}{2} S_x^2} = S_x^1.
$$

(12)

We can apply this transformation to the Heisenberg hamiltonian and find:

$$
H'_{\Lambda,\beta,h} = U^{-1} H^{\text{heis}}_{\Lambda,\beta,h} U = H^{\text{heis}}_{\Lambda,\beta,0} - h \sum_{x \in \Lambda} S_x^1,
$$

(13)
i.e. the magnetic field is now aligned with the first direction of spin. Since the partition function is positive by definition, it is sufficient to study the sign of

\[ \text{Tr} \prod_{x \in X} S_x^3 e^{-H_{\Lambda,\beta,0}^{\text{heis}}} \]

We use the cyclicity of the trace and the equations above:

\[ \text{Tr} \prod_{x \in X} S_x^3 e^{-H_{\Lambda,\beta,0}^{\text{heis}}} = \text{Tr} U^{-1} \prod_{x \in X} S_x^3 U^{-1} e^{-H_{\Lambda,\beta,0}^{\text{heis}}} \]

\[ = \text{Tr} \prod_{x \in X} S_x^1 e^{-U^{-1} H_{\Lambda,\beta,0}^{\text{heis}} U} = \text{Tr} \prod_{x \in X} S_x^1 e^{-H_{\Lambda,\beta,0}'} \]  

(14)

Since \( S^1 \) has non negative matrix elements, \( \text{Tr} \prod_{x \in A} S_x^1 e^{-H_{\Lambda,\beta,0}'} \) is nonnegative because it is the trace of an operator with nonnegative elements. The result is thus proved.

\[ \text{Exercise 4.2.} \] Let \( H_{\Lambda,\beta,0} = -\beta \sum_{(x,y) \in \mathcal{E}_\Lambda} \left( S_x^1 S_y^1 + u S_x^2 S_y^2 + S_x^3 S_y^3 \right) \) where \( u \in [0,1] \) is a fixed parameter. Find operators \( b_{xy} \) that satisfy Eq. (4.3) of the lecture notes and such that the Gibbs state \( \langle \cdot \rangle'_{\Lambda,\beta} \) with hamiltonian \( H_{\Lambda,\beta}' = -\sum_{(x,y) \in \mathcal{E}_{\Lambda,\beta}} b_{xy} \) has identical spin correlations i.e. \( \langle S_x^3 S_y^3 \rangle_{\Lambda,\beta,0} = \langle S_x^3 S_y^3 \rangle'_{\Lambda,\beta} \).

**Proof.** Notice that:

\[ H_{\Lambda,\beta,0} = -\beta \sum_{(x,y) \in \mathcal{E}_\Lambda} u \left( S_x^1 S_y^1 + S_x^2 S_y^2 + S_x^3 S_y^3 \right) + (1-u)S_x^1 S_y^1 + S_x^3 S_y^3 \]

\[ = -\beta \sum_{(x,y) \in \mathcal{E}_\Lambda} \frac{u}{2} \left( S_x^+ S_y^- + S_x^- S_y^+ \right) + (1-u)S_x^1 S_y^1 + S_x^3 S_y^3 \]  

(15)

where \( S^\pm = S^1 \pm i S^2 \). Define

\[ b_{xy} = \beta \left( \frac{u}{2} \left( S_x^+ S_y^- + S_x^- S_y^+ \right) + (1-u)S_x^1 S_y^1 + S_x^3 S_y^3 + \frac{1}{4} \mathbb{1} \right) \]  

(16)

Notice that \( b_{xy} \) applied to a state \( \omega \) acts only on the spins at sites \( x \) and \( y \). Moreover, since \( S_x^\pm, S_x^1 \) and \( S_x^3 S_y^3 + \frac{1}{4} \mathbb{1} \) have nonnegative matrix elements for any \( x, y \in \Lambda \). The conditions in Eq. (4.3) of the lecture notes are thus satisfied. Notice that \( H_{\Lambda,\beta}' \) so defined differs from \( H_{\Lambda,\beta,0} \) only by a constant, which does not affect the Gibbs state. The result is thus proved.

**Exercise A.1 (i).** Show that \( \lim_{p \to \infty} \|A\|_p = \|A\|_\infty \)

**Proof.** Recall the singular value decomposition: for any square \( n \times n \) matrix \( A \) there exist \( V \) and \( W \) unitary matrices such that \( A = V D W^* \) with \( D = \text{diag} (\sigma_1(A), \sigma_2(A), \ldots, \sigma_n(A)) \),
with \( \sigma_1(A) \geq \sigma_2(A) \geq \cdots \geq \sigma_n(A) \geq 0 \). The \( \sigma_i(A) \) are called “singular values” – in the case \( A \) is hermitian they are the absolute values of eigenvalues. Notice that

\[
\|A\|_\infty = \sup_{x: \|x\|=1} (x, A^*Ax)^{\frac{1}{2}} = \sup_{x: \|x\|=1} (x, V^*D^2Vx)^{\frac{1}{2}} = \sigma_1(A),
\]

due to the unitarity of \( V \) and \( W \). Moreover, by the cyclicity of trace:

\[
\|A\|_p = \left( \sum_i \sigma_i(A)^p \right)^{\frac{1}{p}}.
\]

Notice that

\[
\lim_{p \to \infty} A_p = \sigma_1(A) \left( 1 + \sum_{i \geq 2} \left( \frac{\sigma_i(A)}{\sigma_1(A)} \right)^p \right)^{\frac{1}{p}} = \sigma_1(A),
\]

since \( \frac{\sigma_i(A)}{\sigma_1(A)} < 1 \) for all \( i \geq 2 \).

**Exercise A.1 (ii).** Show that \( \|A\|_p \) is a norm for all \( 1 \leq p \leq \infty \).

*Proof.* To prove this statement, we use extensively Eq. (18). The properties we need to show are:

1. \( \|A\|_p \geq 0 \) for any \( A \) square matrix,
2. \( \|A\|_p = 0 \) if and only if \( A = 0 \),
3. \( \|\alpha A\|_p = |\alpha|\|A\|_p \) for all complex \( \alpha \),
4. \( \|A + B\|_p \leq \|A\|_p + \|B\|_p \).

Properties 1. and 2. are straightforward from Eq. (18). Property 3. can be proved explicitely:

\[
\|\alpha A\|_p = (\text{Tr} \left( (\alpha A)^*(\alpha A) \right)^{\frac{p}{2}})^{\frac{1}{p}} = |\alpha|\|A\|_p
\]

for any \( \alpha \in \mathbb{C} \) and any square matrix \( A \). To prove the triangular inequality, we need the following Lemma (its proof can be found, for instance, in *Topics in Matrix Analysis* by R.A. Horn and C.R. Johnson, Cambridge University Press (1991), Lemma 3.3.8):
Lemma 1. Let \( x_1 \geq x_2 \geq \cdots \geq x_n \) and \( y_1 \geq y_2 \geq \cdots \geq y_n \), with \( x_i, y_i \in \mathbb{R} \) and \( n \in \mathbb{N} \) such that:
\[
\sum_{i=1}^{k} x_i \leq \sum_{i=1}^{k} y_i, \quad k \in \{1, \ldots, n\}.
\]
Let \( f \) be an increasing and convex function real valued function. Then, \( f(x_1) \geq \cdots \geq f(x_n) \), \( f(y_1) \geq \cdots \geq f(y_n) \) and
\[
\sum_{i=1}^{k} f(x_i) \leq \sum_{i=1}^{k} f(y_i), \quad k \in \{1, \ldots, n\}.
\]
Firstly, we prove the triangular inequality for \( \| \cdot \|_1 \). Notice that for \( A \) with singular value decomposition \( A = V D W^* \):
\[
\sup_{C : \|C\|_\infty = 1} \text{Tr} \ AC = \sup_{C : \|C\|_\infty = 1} \text{Tr} \ DW^* CV. \tag{21}
\]
Due to the unitarity of \( W \) and \( V \), \( \|W^* CV\|_\infty = \|C\|_\infty \), so:
\[
\sup_{C : \|C\|_\infty = 1} \text{Tr} \ AC = \sup_{C : \|C\|_\infty = 1} \text{Tr} \ DC = \|A\|_1. \tag{22}
\]
Then:
\[
\|A + B\|_1 = \sup_{C : \|C\|_\infty = 1} \text{Tr} \ (A + B)C \leq \sup_{C : \|C\|_\infty = 1} \text{Tr} \ AC + \sup_{C : \|C\|_\infty = 1} \text{Tr} \ BC \tag{23}
\]
\[
= \|A\|_1 + \|B\|_1.
\]
Let us now extend this result to any value of \( p \). Notice that for \( \| \cdot \|_1 \) the triangular inequality is equivalent to
\[
\sum_i \sigma_i(A + B) \leq \sum_i \sigma_i(A) + \sigma_i(B). \tag{24}
\]
By Eq. \[(24)\] and the Lemma above:
\[
\|A + B\|_p = \left( \sum_i (\sigma_i(A + B))^p \right)^{\frac{1}{p}} \leq \left( \sum_i (\sigma_i(A) + \sigma_i(B))^p \right)^{\frac{1}{p}} \leq \left( \sum_i \sigma_i(A)^p \right)^{\frac{1}{p}} + \left( \sum_i \sigma_i(B)^p \right)^{\frac{1}{p}} = \|A\|_p + \|B\|_p. \tag{25}
\]
The last line of the expression above comes from Minkowski inequality: for any \((x_1, \ldots, x_n), (y_1, \ldots, y_n) \in \mathbb{R}^n\) and \( p \geq 1 \)
\[
\left( \sum_{i=1}^{n} |x_i + y_i|^p \right)^{\frac{1}{p}} \leq \left( \sum_{i=1}^{n} |x_i|^p \right)^{\frac{1}{p}} + \left( \sum_{i=1}^{n} |y_i|^p \right)^{\frac{1}{p}}. \tag{26}
\]
\( \square \)
**Exercise A.1 (iii).** Use Hölder inequality to show that $\|A\|_p$ is submultiplicative, that is, $\|AB\|_p \leq \|A\|_p \|B\|_p$.

**Proof.** Firstly, let us show that $\|A\|_p$ is nonincreasing in $p$ for any square matrix $A$. Since $\|A\|_p$ is just the $\ell_p$-norm of the vector of singular values of $A$, it is sufficient to prove that the $\ell_p$-norm is nonincreasing in $p$. Let $q \geq p$ and $x = (x_1, \ldots, x_n)$ with $x_i \geq 0$. Assume without loss of generality that $\|x\|_q = 1$ (and by consequence $x_i \leq 1$). It is sufficient to prove

$$\log \|x\|_p \geq \log \|x\|_q = 0. \tag{27}$$

By the concavity of the logarithm:

$$\log \|x\|_p = \frac{1}{p} \log \sum_i x_i^p = \frac{1}{p} \log \sum_i x_i^q x_i^{-(q-p)} \geq \frac{1}{p} \sum_i x_i^q \log x_i^{-(q-p)} = -\frac{q-p}{p} \sum_i x_i^q \log x_i \geq 0. \tag{28}$$

Now, let $p \geq 1$ and $r, r' \geq 1$ such that $\frac{1}{r} + \frac{1}{r'} = \frac{1}{p}$. It follows that $p = \frac{rr'}{r + r'}$, so $p \leq r, r'$. Then, by Hölder inequality and by monotonicity of the $p$-norm:

$$\|AB\|_p \leq \|A\|_r \|B\|_{r'} \leq \|A\|_p \|B\|_p. \tag{29}$$

$\Box$